J. Fluid Mech. (1994), *vol.* 266, *pp.* 243–276

Solitons, solitary waves, and voidage disturbances in gas-fluidized beds

By S. E. HARRIS AND D. G. CRIGHTON

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

(Received 1 February 1993 and in revised form 15 August 1993)

In this paper, we consider the evolution of an initially small voidage disturbance in a gas-fluidized bed. Using a one-dimensional model proposed by Needham & Merkin (1983), Crighton (1991) has shown that weakly nonlinear waves of voidage propagate according to the Korteweg–de Vries equation with perturbation terms which can be either amplifying or dissipative, depending on the sign of a coefficient. Here, we investigate the unstable side of the threshold and examine the growth of a single KdV voidage soliton, following its development through several different regimes. As the size of the soliton increases, KdV remains the leading-order equation for some time, but the perturbation terms change, thereby altering the dependence of the amplitude on time. Eventually the disturbance attains a finite amplitude and corresponds to a fully nonlinear solitary wave solution. This matches back directly onto the KdV soliton and tends exponentially to a limiting size. We interpret the series of large-amplitude localized pulses of voidage formed in this way from initial disturbances as corresponding to the 'voidage slugs' observed in gas fluidization in narrow tubes.

1. Introduction

Fluidized beds are used extensively in industry for processes such as catalytic reactions, which require a good contact area between a fluid and a solid. Particles with diameters ranging from 50 μm to 1 mm are normally used, and these rest on each other on a porous plate in a vertical tube. Fluid is pumped upward through the system and, at a certain velocity, the upward drag of the fluid on the particles balances the gravitational force. Then the particles become mobile and buoyant, and the bed is said to be fluidized. In this state, it exhibits many liquid-like phenomena, such as surface waves.

In gas-fluidized beds, it is possible for uniform expansion to occur over a small range of gas velocities (particularly for fine powders) but at higher flow rates the bed becomes unstable and some of the fluid passes through the bed in the form of bubbles or slugs. This behaviour can also occur in liquid-fluidized beds when the ratio of the density of the solid to that of the liquid is high (e.g. lead shot fluidized with water). In most liquid-fluidized beds, however, although instability is present and can be seen in the form of wavy structures, this does not lead rapidly to bubble formation.

Bubbles are approximately spherically shaped regions of essentially particle-free gas propagating upward. They occur in wide beds, whereas slugs are present in narrow tubes. There are two sorts of slugging, both of which are strongly influenced by the container walls. In the first type, highly deformed bubbles (whose frontal diameters are equal to the size of the tube) rise slowly and displace particles, which flow down at the sides. In the second type, which is common in tubes up to 5 cm in diameter, we have slowly rising bands of particle-deficient regions, with rapid particle raining occurring at the interfaces. More detailed descriptions of these different regimes can be found in Davidson & Harrison (1963) and Zenz (1971). It is with the second type of voidage slug that we are concerned here, through study of a strictly one-dimensional but highly nonlinear model.

The presence of bubbles or slugs in a fluidized bed is undesirable in practical applications, as the efficiency of the process is reduced because of the diminished contact surface between the two phases. Therefore, much interest has been shown in the mathematical modelling of such systems in order to try and understand the mechanisms governing instability. However, the modelling itself is highly controversial, with many authors trying to formulate governing equations using a variety of different assumptions and experimental correlations.

A popular approach, which we shall adopt here, used by Jackson (1963*a*, *b*), Garg & Pritchett (1975), Homsy, El-Kaissy & Didwania (1980), Needham & Merkin (1983), Foscolo & Gibilaro (1984, 1987), Crighton (1991) and others, is that of interacting continua. The point variables are averaged over a volume which is small compared to the whole system, but is large compared to the particle size and spacing. This yields continuum equations for the particles and for the fluid, which are then treated as two interpenetrating one-phase fluids. It is usually helpful to write the equations in terms of a voidage fraction, which is defined to be the volume of fluid in unit volume of the two-phase mixture.

Jackson (1963*a*), Murray (1965), Garg & Pritchett (1975) and Homsy *et al.* (1980) have all carried out linear instability analyses of a small disturbance superimposed on a uniformly fluidized bed. Jackson (1963*a*) predicts greater growth rates in gas-fluidized beds than in liquid ones but both he and Murray (1965) find the bed to be unstable in all cases. This disagrees with observations, and Garg & Pritchett (1975) show the importance of including a 'particle pressure' which has a stabilizing influence. The linear instability analysis does not guarantee that the perturbation will grow into well-formed bubbles or slugs, as once the perturbation amplitude is large enough, nonlinear effects must be taken into account, while further linear mechanisms also come into play, and tend themselves to *disperse* voidage concentrations. This is supported by the experimental evidence of El-Kaissy & Homsy (1976) who have made careful visual and quantitative measurements in liquid-fluidized beds showing that voidage waves initially show an exponential increase in amplitude, but that this growth is not maintained, and eventually equilibrium values of amplitude and velocity are reached. The qualitative results are expected to be similar for gas-fluidized beds.

A few studies have examined nonlinear effects, although the inclusion of the nonlinear terms presents a severe impediment to analysis. Fanucci, Ness & Yen (1979) consider nonlinear continuum conservation equations for mass and momentum, but they neglect internal friction to give a hyperbolic system, which can then be solved by using the method of characteristics. Their analysis is limited to an initially sinusoidal perturbation and they show that a shock wave can form, which they take to be a bubble front. Verloop & Heertjes (1970) also consider shock waves and bubbles to be equivalent and examine shock wave formation as a criterion for bubbling behaviour. However, comparison between experiment and theory by Homsy *et al.* (1980) suggests that internal friction effects may have a significant influence on small-amplitude waves, so that one needs to include 'particle viscosity' effects in the model equations. Sasa & Hayakawa (1992) derive a Korteweg–de Vries (KdV) equation for weakly unstable fluidized beds when the particle viscosity is significant. Komatsu & Hayakawa (1993) examine the Sasa & Hayakawa (1992) model numerically in one dimension and observe soliton-like behaviour in that an initial condition corresponding to the presence of two KdV solitons is found to evolve through propagation and interaction of the two pulses essentially as prescribed for the solitons of the KdV equation. They also derive a perturbed KdV equation which includes a weak dependence on a transverse coordinate. Liu (1982) examines the linear theory in terms of wave hierarchies and derives a linearized Burgers–Korteweg–de Vries equation for the voidage fraction. In a later paper (Liu 1983), weak nonlinear effects are taken into account and it is shown that, for supercritical disturbances, amplitude and velocity equilibration is possible (after truncation of an infinite set of coupled nonlinear equations).

Needham & Merkin (1983) analyse both the linearized equations for their gas-fluidized bed model and also a nonlinear model equation, the latter showing from limited numerical evidence that voidage fronts will develop. However, there is some doubt as to the validity of their nonlinear equation, over which terms have been retained and which have been neglected. Instead, Crighton (1991) derives a Burgers–Korteweg–de Vries equation (where the Burgers term is the leading-order perturbation) for the voidage disturbance to a homogeneous bed and this balances the nonlinear perturbations with the dispersive and small dissipative (amplifying) effects. The stable Burgers–KdV equation has also been derived by Kurdyumov & Sergeev (1987) for small perturbations to a uniformly fluidized bed, but Crighton (1991) shows that the coefficient of the dissipative term can take either sign. In the similar problem of weakly nonlinear waves in suspensions of particles in fluids, Kluwick (1983, 1991) has derived the Burgers–KdV equation and what he terms the 'modified Burgers–KdV equation', which has a cubic nonlinear term as well as a quadratic one. The latter equation is appropriate near the region where the coefficient of the quadratic term vanishes and in both cases the 'dissipative term' can be of either sign. In all these situations, the interest lies on the unstable side of the threshold, as this is where growth occurs and where the characteristic behaviour is not well understood at present.

A recent paper by Batchelor (1988) must be mentioned too. It considers both fluidized beds and the hindered settling of particles in a liquid, which Batchelor argues are essentially the same phenomena, but just referred to different axes. In hindered settling, particles are dispersed throughout the fluid in a vertical tube and fall under the influence of gravity. However, the drag coefficient of each particle is dependent on the particle concentration so that if there are more particles present, then the drag on each one is higher. Thus the rate of settling is lower than might be expected. Batchelor predicts kinematic waves with damping due to the gradient diffusivity of the particles for small Froude number F, with damped dynamic waves at large Froude number. However, if the diffusivity is assumed to be $O(F^2)$ where $F \ll 1$, which is realistic for fluidized beds, then instead of a damped kinematic wave equation, we can again derive the Burgers–KdV equation with the possibility of amplification. Thus this appears to be a canonical equation for the propagation of small voidage disturbances in a uniformly fluidized bed.

Here we will use the one-dimensional Needham & Merkin (1983) model, with a modification to the particle pressure, and present the derivation of the Burgers–KdV equation by Crighton (1991), which is then used as the initial basis for the work in this paper. The leading-order equation is KdV and so any arbitrary initial disturbance will break up into a finite number of solitons which are permanently separated after a short time. Thus it is permissible to just examine the behaviour of an isolated soliton and so, as a suitable initial condition, we begin with a single KdV soliton. The soliton is of greater voidage than the uniform state and rises through the bed at constant velocity and unchanged in shape. If we include the small amplification effects, then the soliton will grow and there is the possibility of a small initial voidage non-uniformity growing to become an $O(1)$ disturbance to the background voidage fraction.

To find a solution to the perturbed KdV equation, we apply a multiple-scales analysis, which allows the arbitrary constant defining the soliton size and speed to vary on a longer timescale. This yields a secularity condition, necessary to suppress resonant terms in the expansion, showing that the *amplitude and velocity have a finite-time singularity* on this new timescale. Physically, this is of course impossible as the voidage fraction must remain less than unity, which is the pure fluid limit, but the presence of a finite-time singularity may be taken to indicate that the initially weakly nonlinear voidage soliton will eventually grow into a large-amplitude wave.

The approximations required to derive the KdV equation with Burgers perturbation terms eventually cease to be valid because the soliton amplitude is growing. If we rescale the original set of equations close to the finite-time singularity, then other terms become of the same order of magnitude as the previous perturbation terms, and so we must retain these as well. In fact there is a whole sequence of such changes as the original perturbation terms become smaller in magnitude relative to the other terms and are exchanged in the asymptotic sequence. Some of these higher-order terms contribute to the secularity condition, and alter the way in which the amplitude of the soliton grows in time, weakening the singularity.

Eventually there is a fundamental change in solution when the soliton perturbation finally becomes $O(1)$ and at this time the original set of equations must be rescaled again to find new, *fully nonlinear* equations for the voidage fraction and the particle velocity. The new equations admit *solitary wave* solutions for the voidage fraction at leading order, although these cannot be expressed in closed form and must be integrated numerically. The dependence of the velocity on the amplitude of the solitary wave can also be written down in terms of quadratures. If a multiple-scales analysis is applied to the solitary wave, allowing its velocity (and hence its amplitude) to vary slowly, then an expression can be found for the secularity condition. An examination of this when the amplitude of the solitary wave is small shows that it reduces to the last secularity condition found for the growing soliton. The method of matched asymptotic expansions then indicates that the soliton and the solitary wave are indeed matched directly without the need for an intermediate, time-dependent region. It is also shown that although the amplitude of the solitary wave is still increasing, there is a limiting value, with the amplitude tending exponentially to this maximum value. Thus, the analysis has been successful in following the development from an infinitesimal perturbation on a uniform bed right through to a fully nonlinear $O(1)$ wave.

2. Basic formulation

Here we will use the one-dimensional model of Needham & Merkin (1983), which is appropriate for gas-fluidized beds in narrow tubes. For a two-phase flow model we require two mass conservation equations: one each for the particles and the fluid, since the particles do not move with the fluid. It is convenient to express the equations in terms of the voidage fraction ϕ. (Note that in some papers, e.g. Batchelor 1988, ϕ is used to mean the particle volume fraction, which is $1-\phi$ here.) We require the voidage to lie between 0 and 1, i.e. the pure solid and pure fluid limits, but in fact the voidage fraction cannot be smaller than approximately 0.26, which is the volume left unfilled by spherical particles in cubic close-packing mode.

Assuming that both phases are incompressible, we have

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}(\phi u) = 0 \tag{2.1}$$

and

$$-\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}([1-\phi]v) = 0, \tag{2.2}$$

for the fluid and particles, respectively, where t is time, x is the coordinate vertically upward, u the fluid velocity and v the particle velocity. These two equations have a first integral

$$\phi u+(1-\phi)v = Q(t), \tag{2.3}$$

which can be used in place of one of the mass conservation equations when closing the system and its shows that the net volume flux across all horizontal planes is the same at any given time.

We also require two momentum equations and, following Crighton (1991), we write the particle momentum equation in the form

$$\rho_s(1-\phi)\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right) = B(\phi)(u-v)-\rho_s(1-\phi)g+\frac{\partial S}{\partial x}, \tag{2.4}$$

where the terms on the left-hand side represent particle inertia, the first term on the right-hand side is the drag force of the fluid on the particles, which is modelled by a drag coefficient $B(\phi)$ multiplied by the relative velocity $(u-v)$ of the two phases, the second term is the gravitational force, and interparticle effects are assumed to be representable as the gradient of a stress S. These interparticle effects arise from short-range forces, including barrier forces which prevent a particle from occupying a site already occupied by another particle. (For a more detailed discussion of the nature of S and of the microscopic phenomena which contribute to S, see Batchelor 1988.)

In a similar way, we can write down a momentum equation for the fluid, namely

$$\rho_f\phi\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right) = -B(\phi)(u-v)-\rho_f\phi g-\frac{\partial p_f}{\partial x}+\mu_f\frac{\partial^2 u}{\partial x^2}. \tag{2.5}$$

We have omitted the virtual mass term from (2.5), which is retained in other models such as that of Auzerais, Jackson & Russel (1988) for the transient settling of particles, but here we are interested solely in gas-fluidized beds and so this approximation is justified. The assumption that $\rho_f \ll \rho_s$ for gas–solid systems leads to a simplification, as a comparison of (2.5) with (2.4) shows that we can neglect the fluid inertia terms on the left-hand side as well as the gravitational term. The fluid viscosity μ_f is small for a typical gas-fluidized bed, so the explicit appearance of this term may also be neglected, although fluid viscosity effects are important in the drag force, but these are incorporated into the drag coefficient term $B(\phi)$. Thus the fluid momentum equation reduces to

$$\partial p_f/\partial x = -B(\phi)(u-v), \tag{2.6}$$

and hence just determines the fluid pressure p_f in terms of ϕ, u, v, which must be found from the other three equations. Therefore, we can ignore this equation when solving the system.

To close the system of equations, we must now make some assumptions and hypotheses about the dependence of the drag coefficient B on the voidage fraction ϕ and also about the form taken by the stress S. We will assume that the stress can be decomposed into two constituent elements, one of which we will call a 'particle pressure' p_s, and we shall assume that the second part of the stress depends on the velocity gradient field through a 'particle viscosity' μ_s, in the same way as for a one-phase Newtonian fluid. Thus we can write the stress as

$$S = -p_s + \mu_s \partial v/\partial x. \tag{2.7}$$

Instead of a particle pressure, Batchelor (1988) has a term involving the diffusivity and he also uses a slightly different form for the second term in the stress which essentially entails making the viscosity a function of voidage, vanishing as $\phi \to 1$. However, we will retain the constant-μ_s assumption for the first approximation as this is certainly applicable for small departures from the uniform state.

A suitable form for $B(\phi)$ is given by Needham & Merkin (1983) and Göz (1992) as

$$B(\phi) = \frac{1-\phi}{\phi^n}\frac{D_0}{V_p}, \tag{2.8}$$

where V_p is the volume of a single particle, D_0 is the Stokes drag on an isolated sphere and the index n lies in the range 3–4. This form is chosen so that the uniform state satisfies the Richardson–Zaki relation, which is an experimental correlation (Richardson 1971).

Drew & Segal (1971), Jackson (1971) and Homsy *et al.* (1980) all suggest that the particle pressure p_s is a monotonic decreasing function of the voidage, and Needham & Merkin (1983) assume a linear dependence and impose the condition that the particle pressure must vanish as $\phi \to 1$, which is the pure fluid limit. However, we also require the pressure to prevent the voidage decreasing below the close-packing limit and therefore we suggest the form

$$p_s(\phi) = P_s \frac{1-\phi}{\phi - \phi_{cp}}, \tag{2.9}$$

where P_s is a positive constant and ϕ_{cp} is the voidage at close packing. This form allows a stable regime for a range of $\phi > \phi_{cp}$, unlike the corresponding expression used by Needham & Merkin (1983). More complicated expressions for the dependence of the pressure on voidage are also possible but these have little effect on the analysis.

The equations now form a closed system and they have the simple solution of uniform fluidization:

$$\left.\begin{array}{c} u = U_0, \quad v = 0, \quad \phi = \phi_0, \\ Q = U_0\phi_0, \quad B(\phi_0)\,U_0 = (1-\phi_0)\rho_s g, \quad U_0 = U_t \phi_0^n, \end{array}\right\} \tag{2.10}$$

in which the voidage fraction and gas velocity are constant and the particles have no mean vertical velocity. The last of these expressions is the Richardson–Zaki relation where U_t is the terminal free-fall velocity of an isolated particle in static fluid. If we introduce a localized voidage disturbance of lengthscale h, we can then non-dimensionalize the equations, setting

$$\left.\begin{array}{c} u = U_0\tilde{u}, \quad v = U_0\tilde{v}, \quad x = h\tilde{x}, \\ t = \dfrac{h}{U_0}\tilde{t}, \quad p_s(\phi) = \rho_s U_0^2 \tilde{p}_s(\phi), \quad B(\phi) = \dfrac{\rho_s g}{U_0}\tilde{B}(\phi). \end{array}\right\} \tag{2.11}$$

On dropping the tildes from the dimensionless variables and using (2.3) to eliminate u from the particle momentum equation, we have the system

$$-\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}([1-\phi]v)=0, \tag{2.12}$$

$$(1-\phi)\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=\frac{1-\phi}{F^2}\left(\frac{\phi_0}{\phi}\right)^{n+1}\left(1-\frac{v}{\phi_0}\right)-\frac{1-\phi}{F^2}-p_s'(\phi)\frac{\partial \phi}{\partial x}+\frac{1}{R}\frac{\partial^2 v}{\partial x^2}, \tag{2.13}$$

where $F^2 = U_0^2/gh$ is the square of a Froude number and $R = \rho_s U_0 h/\mu_s$ is a particle-phase Reynolds number. Note that these are different from the Reynolds and Froude numbers used by Batchelor (1988). These equations define ϕ and v as functions of x, t and constitute a particle bed model, as we are just working with the particle mass and momentum conservation equations.

Some experimental results for typical gas-fluidized beds are given by Schügerl (1971). For glass beads with diameters of the order of 50 μm, the minimum interstitial fluidization velocity is approximately 5.7 cm s^{-1}, with an effective kinematic bed viscosity of 1.9 cm^2 s^{-1}. The experiments by El-Kaissy & Homsy (1976) suggest that voidage waves have a width of 1 cm and so we will take this as our disturbance lengthscale h. Thus $F^2 = 0.03$, $R = 3$ and so the scalings $R = O(1)$, and $F^2 \ll 1$ are appropriate, and we will use them in the next section. This is in no way to imply that other parts of the (R, F) parameter space are without possible physical interest. However, our consideration of the limit $F \to 0$, $R = O(1)$ seems appropriate for correspondence with some laboratory-scale experiments and will be seen to lead to the emergence of large-scale orderly structure in the form of sequences of permanent bands or slugs of high voidage.

3. Linear and weakly nonlinear instability

In the light of the experimental evidence given by El-Kaissy & Homsy (1976) that the breakup of voidage waves is followed by motion which suggests bubble or slug formation, it seems appropriate to adopt the approach of many other authors in looking at the stability of the uniform state and finding under what conditions voidage waves grow. Thus, we first perform a linear perturbation analysis by letting

$$v = v', \quad \phi = \phi_0 + \phi', \tag{3.1}$$

where v' and ϕ' are small quantities, and substitute these into (2.12) and (2.13). Retaining only the linear terms in the perturbation variables, we have

$$F^2\frac{\partial^2 \phi'}{\partial t^2} = -\frac{\alpha_0}{\phi_0}\frac{\partial \phi'}{\partial x}-\frac{1}{\phi_0}\frac{\partial \phi'}{\partial t}+F^2 P_0\frac{\partial^2 \phi'}{\partial x^2}+\frac{F^2}{R(1-\phi_0)}\frac{\partial^3 \phi'}{\partial x^2 \partial t}, \tag{3.2}$$

with
$$\alpha_0 = (n+1)(1-\phi_0), \quad P_0 = -p_s'(\phi_0) = P_s\frac{1-\phi_{cp}}{(\phi_0-\phi_{cp})^2} > 0. \tag{3.3}$$

If we assume P_0 is $O(1)$ and use $F^2 \ll 1$, then examination of (3.2) indicates that over scales where x and t are both $O(1)$, the appropriate equation for ϕ' is

$$\frac{\partial \phi'}{\partial t}+\alpha_0\frac{\partial \phi'}{\partial x} = 0, \tag{3.4}$$

so any disturbance propagates upwards, unchanged in form and with the same constant speed α_0.

Over long times, we can define a new time variable $\tau = F^2 t$ and when this is $O(1)$, equation (3.2) becomes

$$\frac{\partial \phi'}{\partial \tau}+\gamma_0\frac{\partial^3\phi'}{\partial X^3} = \phi_0(P_0-\alpha_0^2)\frac{\partial^2\phi'}{\partial X^2}+O(F^2), \tag{3.5}$$

where $X = x-\alpha_0 t$ and $\gamma_0 = (n+1)\,\phi_0/R$. The corresponding dispersion relation can be found by substituting

$$\phi' = A\,\mathrm{e}^{\mathrm{i}(kX-\omega\tau)} \tag{3.6}$$

into (3.5) to give

$$\omega(k) = -\gamma_0 k^3-\mathrm{i}\phi_0(P_0-\alpha_0^2)\,k^2+O(F^2). \tag{3.7}$$

Thus on this long timescale, the particle viscosity has a dispersive effect, whilst the particle pressure is diffusive, and stabilizing if $P_0 \geqslant \alpha_0^2$. These linear diffusive and dispersive mechanisms naturally cause any initial concentration of voidage or particles to be relieved.

The stability condition $P_0 \geqslant \alpha_0^2$ was derived in the limit $F^2 \ll 1$, but in fact it applies for all wavenumbers at any value of F, as indicated by Needham & Merkin (1983). However if $P_0 < \alpha_0^2$, then for some k_c, modes with wavenumbers $k \geqslant k_c$ are stable and those with $0 < k < k_c$ are unstable (see Appendix A). We note that $P_0 = \alpha_0^2$ gives

$$\frac{P_s(1-\phi_{cp})}{(n+1)^2} = (\phi_0-\phi_{cp})^2\,(1-\phi_0)^2, \tag{3.8}$$

from (3.3) and so for $P_0 < \alpha_0^2$ to hold for some ϕ_0 in the range $\phi_{cp} < \phi_0 < 1$, we must have

$$P_s < \tfrac{1}{16}(n+1)^2\,(1-\phi_{cp})^3. \tag{3.9}$$

We are interested in the marginally unstable case, so we consider $\alpha_0^2-P_0$ to be small and positive and set

$$\alpha_0^2-P_0 = F\frac{\delta_0}{\phi_0}. \tag{3.10}$$

Then for wavenumbers such that $k \ll k_c$, the dispersion relation (3.7) gives the leading-order behaviour for the real and imaginary parts of ω. For larger wavenumbers where $k \sim k_c$, a rescaling is required and gives rise to a fourth derivative on the right-hand side of (3.5) to balance the amplification term (see Appendix A). Here, we will just be concerned with the $k \ll k_c$ approximation.

In the unstable situation $P_0 < \alpha_0^2$, disturbances of small but finite amplitude will grow and eventually nonlinear effects must be included. We note that numerically there will be problems because in (3.5) any short-wave components will grow much more rapidly than the longer wavelengths so that the problem appears ill-posed. However, this is spurious and is due to the neglected terms such as a fourth derivative which control the growth. (See (A 18) for the specific form.) It is only in the *fully nonlinear* problem that equilibration of the long wavelength occurs and so any numerical solution must be of the *full equations* (2.12) and (2.13).

Needham & Merkin (1983) derive a nonlinear wave equation which has shock solutions between states of uniform voidage, but Crighton (1991) disagrees with their equation and its formulation. Instead, he balances quadratically nonlinear perturbations to the linear waves with weak dispersive effects. Here, we simplify the analysis given by Crighton (1991) by using the system (2.12) and (2.13) and the scaled representations

$$v = F^2 v', \quad \phi = \phi_0 + F^2 \phi' \tag{3.11}$$

for v and ϕ, together with

$$X = x - \alpha_0 t, \quad \tau = F^2 t. \tag{3.12}$$

This gives the set of equations

$$\alpha_0 \frac{\partial \phi'}{\partial X} + (1-\phi_0)\frac{\partial v'}{\partial X} + W_1 = 0, \tag{3.13}$$

$$-\frac{\alpha_0}{\phi_0}\phi' - (1-\phi_0)\frac{v'}{\phi_0} + W_2 = 0, \tag{3.14}$$

where

$$W_1 = -F^2\left[\frac{\partial \phi'}{\partial \tau} + \frac{\partial}{\partial X}(\phi' v')\right], \tag{3.15}$$

and

$$W_2 = F^2\left[\frac{\alpha_0(n+2)}{2\phi_0^2}\phi'^2 - \frac{\alpha_0}{\phi_0^2}\phi' v' + \frac{n+1}{\phi_0}\phi'^2 + \frac{\phi' v'}{\phi_0}\right] + F^2\left[P_0 \phi'_X + \alpha_0(1-\phi_0) v'_X + \frac{1}{R} v'_{XX}\right] + O(F^4). \tag{3.16}$$

We can now eliminate the linear $O(1)$ terms, to give

$$W_1 + \phi_0 \partial W_2 / \partial X = 0. \tag{3.17}$$

The $O(1)$ approximation is given by putting the $W_i = 0$ in (3.13) and (3.14), to yield

$$v' = -(n+1)\phi', \tag{3.18}$$

which can then be used in (3.17) to give the following Burgers–Korteweg–de Vries equation for the voidage perturbation ϕ':

$$\frac{\partial \phi'}{\partial \tau} + \beta_0 \phi' \frac{\partial \phi'}{\partial X} + \gamma_0 \frac{\partial^3 \phi'}{\partial X^3} = -F\delta_0 \frac{\partial^2 \phi'}{\partial X^2} + O(F^2), \tag{3.19}$$

where

$$\beta_0 = -(n+1)\left(2 + n - \frac{n}{\phi_0}\right), \quad \gamma_0 = (n+1)\frac{\phi_0}{R}, \quad F\delta_0 = \phi_0(\alpha_0^2 - P_0), \tag{3.20}$$

so that β_0, γ_0 and δ_0 are $O(1)$ constants.

At leading order, we have the KdV equation, which has the well-known single soliton solution

$$\phi' = \frac{12\gamma_0}{\beta_0}\kappa^2 \operatorname{sech}^2[\kappa(x - x_0 - \alpha_0 t - 4\gamma_0 \kappa^2 \tau)], \tag{3.21}$$

where κ and x_0 are arbitrary constants. For $\beta_0 > 0$, we have a soliton of greater voidage than the uniform state, propagating upwards with the constant velocity $c = \alpha_0 + 4\gamma_0 \kappa^2 F^2$. However, if $\beta_0 < 0$, we would have solitons of lower voidage, i.e. a denser fluid-plus-particles mixture, rising through the bed, which certainly appears unreasonable if the bed was tightly packed initially. In this case, moreover, regions of initially higher voidage would *disperse*, according to standard KdV theory (see Drazin

& Johnson 1989). Normally, for a bed of spherical particles, ϕ_0 lies in the range 0.4–0.5 (Leva 1959) and thus, if the index n from the Richardson–Zaki relation is between 3 and 4, we will have

$$\phi_0 < n/(n+2), \tag{3.22}$$

which corresponds to β_0 positive. Therefore, it seems necessary to confine ourselves only to analysing the solution when $\beta_0 > 0$, as the only physically realizable situation.

For the case $P_0 > \alpha_0^2$, i.e. $\delta_0 < 0$ in (3.19), we have the Burgers–Korteweg–de Vries equation, with damping. This equation has been derived by Kuznetsov *et al.* (1978) to describe long-wave perturbations in a liquid with gas bubbles, and by Johnson (1972) to describe the surface profile of an undular bore. In fact, the equation given by Johnson (1972) has the coefficient δ_0 able to change sign for different Froude numbers, but the paper only considers the equation with damping. Monotonic and oscillatory shock solutions are possible between two constant levels, or for small negative δ_0 we can think of (3.19) as a perturbed KdV equation having soliton solutions with slowly decreasing amplitude, as considered by Karpman & Maslov (1977). This obviously refers to the stable side of the threshold, so any small initial disturbance will die out as it propagates upwards.

Thus, we are left with the case $P_0 < \alpha_0^2, \delta_0 > 0$, which yields the unstable Burgers–KdV equation, with amplification instead of damping. Analysis of this will hopefully give insight into the onset of bubbling, or more particularly slugging, and the breakdown of normal fluidization. Of course, our equation for the voidage perturbation was derived under the assumption that the perturbation magnitude was small, so the perturbed KdV model equation fails when this condition is violated. Section 5 will provide the description that replaces this when the perturbations have grown to $O(1)$.

4. Perturbed Korteweg–de Vries solitons

We begin with the perturbed KdV equation

$$\frac{\partial \phi'}{\partial \tau} + \beta_0 \phi' \frac{\partial \phi'}{\partial X} + \gamma_0 \frac{\partial^3 \phi'}{\partial X^3} = -F\delta_0 \frac{\partial^2 \phi'}{\partial X^2} + O(F^2), \tag{4.1}$$

where $\beta_0, \gamma_0, \delta_0 > 0$ and $0 < F \ll 1$, and we look for an asymptotic expansion for ϕ' in powers of F. Then the leading-order solution satisfies the KdV equation and we can choose a single KdV soliton as our initial condition. However, the second-order solution has resonant terms which would lead to the expansion breaking down at times $\tau = O(F^{-1})$. Therefore, to obtain a solution valid at this time, we need to apply a multiple-scales analysis (Leibovich & Seebass 1974; Nayfeh 1973) to equation (4.1), whereby dependences on slow timescales of the form $F^k\tau$ are introduced into the asymptotic expansion for ϕ', in order to eliminate the secular or resonant terms. Here we introduce a single slow timescale,

$$T = F\tau, \tag{4.2}$$

which is the timescale on which a voidage soliton changes as it rises. Therefore, we look for an expansion for ϕ' in the form

$$\phi' = \phi_1(X, \tau, T) + F\phi_2(X, \tau, T) + \dots, \tag{4.3}$$

replacing (4.1) by

$$\phi'_\tau + \beta_0 \phi' \phi'_X + \gamma_0 \phi'_{XXX} = -F\delta_0 \phi'_{XX} - F\phi'_T. \tag{4.4}$$

Thus, ϕ_1 still satisfies the KdV equation, but the inclusion of the long timescale T allows the 'constants' κ and x_0 from (3.21) to become functions of T. We must also modify the displacement term so that the leading-order solution becomes

$$\phi_1(X,\tau,T) = \frac{12\gamma_0}{\beta_0}\kappa^2(T)\,\mathrm{sech}^2\,[\kappa(T)\,(X-x_0(T)-\xi)], \tag{4.5}$$

where

$$\partial\xi/\partial\tau = 4\gamma_0\,\kappa^2(T). \tag{4.6}$$

Formally, we must write

$$\xi = \frac{4\gamma_0}{F}\int\kappa^2(T)\,\mathrm{d}T, \tag{4.7}$$

rather than the alternative form

$$\xi = 4\gamma_0\,\kappa^2(T)\,\tau \tag{4.8}$$

(which also satisfies the condition (4.6) if τ and T are treated as independent variables) because this latter expression introduces spurious resonant terms into the multiple-scales expansion due to the multiplication of the different timescales. Our approach is also used implicitly by Ablowitz & Segur (1981, p. 252) and Knickerbocker & Newell (1980).

To find the dependence of κ and x_0 on T, we need to consider higher-order terms in the asymptotic expansion. The equation for ϕ_2 in terms of ϕ_1 can be obtained by substituting (4.3) into (4.4) to give

$$\phi_{2\tau}+\beta_0(\phi_2\,\phi_1)_X+\gamma_0\,\phi_{2XXX} = -\delta_0\,\phi_{1XX}-\phi_{1T}. \tag{4.9}$$

If we change variables to

$$\eta = X-x_0-\xi, \quad s=\tau, \tag{4.10}$$

and set

$$\theta = \kappa\eta, \tag{4.11}$$

then

$$\phi_1 = \frac{12\gamma_0}{\beta_0}\kappa^2\,\mathrm{sech}^2\,\theta, \tag{4.12}$$

and (4.9) becomes

$$\phi_{2s}-4\gamma_0\,\kappa^2\phi_{2\eta}+12\gamma_0\,\kappa^2\{\phi_2\,\mathrm{sech}^2\,\theta\}_\eta+\gamma_0\,\phi_{2\eta\eta\eta} = -\frac{24\gamma_0\,\delta_0}{\beta_0}\kappa^4\{2\,\mathrm{sech}^2\,\theta-3\,\mathrm{sech}^4\,\theta\}$$

$$-\frac{24\gamma_0}{\beta_0}\{\kappa\kappa_T\,\mathrm{sech}^2\,\theta-\kappa\kappa_T\,\theta\,\mathrm{sech}^2\,\theta\tanh\theta\}-\frac{24\gamma_0}{\beta_0}\{\kappa^3 x_{0T}\,\mathrm{sech}^2\,\theta\tanh\theta\}. \tag{4.13}$$

The multiple-scales technique is intended to provide a solution valid for $T=O(1)$ and so we seek a quasi-stationary solution for ϕ_2, independent of s. Then there is a solution to (4.13) in the form

$$\phi_2(X,\tau,T) = \chi+\Omega\tanh\theta+(A+B\theta+C\theta^2)\,\mathrm{sech}^2\,\theta\tanh\theta+(D+E\theta)\,\mathrm{sech}^2\,\theta. \tag{4.14}$$

Substituting this expression back into (4.13) yields the consistency conditions

$$C = \frac{3}{2\beta_0}\frac{\kappa_T}{\kappa^2} = -\frac{E}{2}, \quad \Omega = \frac{8}{5}\frac{\delta_0}{\beta_0}\kappa, \quad B=-D, \tag{4.15}$$

$$B-3\chi = -\frac{3}{\beta_0}x_{0T}, \quad \kappa_T = \frac{8}{15}\delta_0\,\kappa^3, \tag{4.16}$$

where x_{0T}, A and B are arbitrary functions of T.

Owing to the original non-dimensionalization, we can impose the initial condition $\kappa(0) = 1$ and integrate up the final equality in (4.16) to give

$$\kappa(T) = \left(\frac{T_0}{T_0 - T}\right)^{\frac{1}{2}}, \tag{4.17}$$

where

$$T_0 = \frac{15}{16\delta_0}. \tag{4.18}$$

This expression for κ shows that there is a *finite-time singularity* at $T = T_0$ and for shorter times κ is a slowly increasing function, so that the soliton is amplified. The finite-time singularity is *necessary* if the weakly nonlinear soliton is to be amplified to a fully nonlinear voidage wave, though of course this singularity will not be present in the final waveform. Equation (4.17) can also be found from global 'energy' conservation,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\infty}^{\infty} \tfrac{1}{2}\phi'^2 \mathrm{d}X + \int_{-\infty}^{\infty} (\beta_0 \phi'^2 \phi'_X + \gamma_0 \phi' \phi'_{XXX})\,\mathrm{d}X = -\int_{-\infty}^{\infty} F\delta_0 \phi' \phi'_{XX}\,\mathrm{d}X, \tag{4.19}$$

using (4.3) and (4.5) in (4.19) and neglecting terms of magnitude $O(F^2)$. Substituting (4.17) into (4.7) gives an equation for ξ, namely

$$\xi = -4\gamma_0 \frac{T_0}{F} \ln\left(1 - \frac{T}{T_0}\right). \tag{4.20}$$

In the solution given by (4.14), the terms involving A and B can be put equal to zero, as they can be absorbed into the first-order solution, ϕ_1, by changing κ and x_0 by an $O(F)$ quantity. Examining the behaviour of ϕ_2 as $\theta \to +\infty$ and imposing the condition that the medium is undisturbed ahead of the soliton, so $\phi_2 \to 0$ as $\theta \to +\infty$, gives

$$\chi = -\frac{8}{5}\frac{\delta_0}{\beta_0}\kappa. \tag{4.21}$$

From (4.16), we then have

$$x_{0T} = -\tfrac{8}{5}\delta_0 \kappa, \tag{4.22}$$

so that

$$x_0(T) = 3\left(1 - \frac{T}{T_0}\right)^{\frac{1}{2}}. \tag{4.23}$$

The expression for ϕ_2 is now given from (4.14) by

$$\phi_2 = \frac{8}{5}\frac{\delta_0}{\beta_0}\kappa(-1 + \tanh\theta + \tfrac{1}{2}\theta^2 \operatorname{sech}^2\theta \tanh\theta - \theta \operatorname{sech}^2\theta). \tag{4.24}$$

The expansion (4.3) for ϕ' is valid for

$$\kappa\theta^2 < O(F^{-1}), \tag{4.25}$$

and while κ is still $O(1)$, the solution holds around the soliton core. To find the solution outside this region, we need to rescale the coordinates and we can no longer use the quasi-stationary assumption, whereby the ϕ_2 term depends on the long timescale T but not on τ. It is well known that in such perturbed KdV problems usually there are spatially non-uniform effects beyond the core, such as a finite trailing shelf and an oscillatory tail (see Knickerbocker & Newell 1980, 1985; Ablowitz & Segur 1981;

Karpman & Maslov 1977, 1978). In fact we can deduce the presence of a shelf below the soliton by noting that

$$\phi_2 \to -\frac{16}{5}\frac{\delta_0}{\beta_0}\kappa \quad \text{as} \quad \theta \to -\infty. \tag{4.26}$$

This shows that the shelf voidage is less than the equilibrium level ϕ_0 and so it corresponds to a *region of higher particle concentration.* The presence of the perturbation term means that the soliton can no longer satisfy its infinity of local conservation laws and, in particular, both the 'mass' and the rate of change of 'energy' laws cannot hold together without additional spatial structure. As we have seen, the soliton itself satisfies global energy conservation to leading order and so the shelf is necessary to help conserve mass. We note that the shelf has a slowly varying amplitude $16F\delta_0\kappa/5\beta_0$ relative to the soliton, but it can contribute an $O(1)$ amount to the mass balance because it grows in length. The finite nature of the shelf and the existence of a tail are not shown here as the adiabatic approximation, in which we just let the soliton parameters vary slowly, proves to be sufficient for present purposes.

The width of the soliton is $O(\kappa^{-1})$, and within this range θ is $O(1)$, so (4.25) is equivalent to

$$T - T_0 < O(F^2), \tag{4.27}$$

which suggests that the solution is valid for the core region until T is within $O(F^2)$ of the finite-time singularity. In fact, the whole perturbation scheme appears to break down at $T - T_0 = O(F^2)$, since then the original perturbation to the bed has now grown to $O(1)$. However, a reordering of terms and a significant change in κ occur before this, as terms neglected in the derivation of the unstable Burgers–KdV equation become of the same order as the retained perturbation terms.

To see this, we need to rescale the coordinates near the finite-time singularity by writing

$$\left.\begin{aligned} -F^{2a}\hat{T} = T_0 - T, \quad \hat{x} = F^{-a}(X - x_0 - \xi), \\ \hat{\kappa}(\hat{T}) = F^a\kappa(T), \quad \frac{\partial\xi}{\partial\tau} = F^{-2a}c_1, \quad c_1 = 4\gamma_0\hat{\kappa}^2(\hat{T}), \end{aligned}\right\} \tag{4.28}$$

where $\hat{T}$, $\hat{x}$ and $\hat{\kappa}$ are $O(1)$ quantities. Substituting these back into (2.12) and (2.13) and now setting $\phi = \phi_0 + F^{2-2a}\phi', v = F^{2-2a}v'$, gives

$$\begin{aligned} -c_1\phi'_{\hat{x}} + \beta_0\phi'\phi'_{\hat{x}} + \gamma_0\phi'_{\hat{x}\hat{x}\hat{x}} = {} & -F^{3a}\phi'_\tau - F^{1+a}[\phi'_{\hat{T}} + \delta_0\phi'_{\hat{x}\hat{x}}] \\ & -F^{2-2a}\left[2c_1\frac{n+1}{\phi_0}\phi'\phi'_{\hat{x}} + \frac{\gamma_0 c_1}{\alpha_0}\phi'_{\hat{x}\hat{x}\hat{x}} + \frac{\gamma_0}{1-\phi_0}(\phi'^2)_{\hat{x}\hat{x}\hat{x}}\right] \\ & -F^{2-2a}\frac{(n+1)(n+2)}{2\phi_0^2}[3\phi_0 - 2n(1-\phi_0)]\,\phi'^2\phi'_{\hat{x}} \\ & -F^{2-a}\phi_0\alpha_0[2c_1\phi'_{\hat{x}\hat{x}} + (n+1)(\phi'^2)_{\hat{x}\hat{x}}] \\ & -F^{2-a}\phi_0 P_1[(\phi'_{\hat{x}})^2 + \phi'\phi'_{\hat{x}\hat{x}}] + o(F^{1+a}, F^{2-a}), \end{aligned} \tag{4.29}$$

where

$$P_1 = p''_s(\phi_0). \tag{4.30}$$

We note that dependence on τ no longer arises at leading order and, as before, we will look for quasi-stationary solutions which depend on $\hat{x}$ and $\hat{T}$, but are independent of τ.

The $O(F^{1+a})$ terms on the right-hand side are those which we retained originally, and which led to soliton amplification according to (4.17). However, it can be seen that when $a = \frac{1}{3}$, the cubic nonlinear terms at $O(F^{2-2a})$ become of the same magnitude as the previous amplifying ones, and so we must now also keep these. As a consequence, although the leading-order solution for the voidage perturbation will still be expressed as a KdV soliton, its amplification may be described by an expression different from that given in (4.17).

To see what happens when $a = \frac{1}{3}$, we note that the expansion for the perturbation ϕ' is given by

$$\phi' = q_1 + F^{\frac{4}{3}} q_2 + o(F^{\frac{4}{3}}),$$

where q_1 satisfies

$$-c_1 q_{1\hat{x}} + \beta_0 q_1 q_{1\hat{x}} + \gamma_0 q_{1\hat{x}\hat{x}\hat{x}} = 0. \tag{4.31}$$

The equation for q_2 can be written as

$$\mathscr{L} q_2 \equiv -c_1 q_{2\hat{x}} + \beta_0 (q_1 q_2)_{\hat{x}} + \gamma_0 q_{2\hat{x}\hat{x}\hat{x}} = G_2, \tag{4.32}$$

where

$$G_2 = -\mu q_1 q_{1\hat{x}} - \varsigma q_1^2 q_{1\hat{x}} - \sigma q_{1\hat{x}\hat{x}\hat{x}} - \rho (q_1^2)_{\hat{x}\hat{x}\hat{x}} - q_{1\hat{T}} - \delta_0 q_{1\hat{x}\hat{x}}, \tag{4.33}$$

with the constants μ, ς, σ, ρ given by

$$\mu = 2c_1 \frac{n+1}{\phi_0}, \quad \sigma = \frac{\gamma_0 c_1}{\alpha_0}, \quad \rho = \frac{\gamma_0}{1-\phi_0}, \tag{4.34}$$

$$\varsigma = \frac{(n+1)(n+2)}{2\phi_0^2} [3\phi_0 - 2n(1-\phi_0)]. \tag{4.35}$$

The operator adjoint to $\mathscr{L}$ is

$$\mathscr{L}^A \equiv c_1 \frac{\partial}{\partial \hat{x}} - \beta_0 q_1 \frac{\partial}{\partial \hat{x}} - \gamma_0 \frac{\partial^3}{\partial \hat{x}^3}, \tag{4.36}$$

so we have

$$\mathscr{L}^A q_1 = 0, \tag{4.37}$$

from (4.31). By the theory of adjoint operators (Courant & Hilbert 1962), the necessary compatibility condition for the existence of q_2 is

$$\int_{-\infty}^{\infty} G_2 q_1 \, \mathrm{d}\hat{x} = 0, \tag{4.38}$$

but by using

$$q_1 = \frac{12\gamma_0}{\beta_0} \hat{\kappa}^2 \operatorname{sech}^2 \hat{\kappa}\hat{x}, \tag{4.39}$$

it can easily be shown that only the last two terms of G_2 give a non-zero contribution to the integral. The result is identical to the energy criterion which we used in (4.19), so there has been no change in the secularity condition. Therefore, we have

$$\hat{\kappa} = F^a \kappa = (-T_0/\hat{T})^{\frac{1}{2}}, \tag{4.40}$$

for $\hat{T} < 0$.

For $\frac{1}{2} > a > \frac{1}{3}$, the asymptotic series just reorders itself, with

$$\phi' = q_1 + F^{2-2a} q_2 + F^{1+a} q_3 + F^{2-a} q_4 + \cdots, \tag{4.41}$$

where q_3 is just the original perturbation solution (4.24), and q_2 is given by

$$q_2 = B\theta \operatorname{sech}^2 \theta \tanh \theta + D \operatorname{sech}^2 \theta + \lambda \operatorname{sech}^4 \theta, \tag{4.42}$$

with B, D and λ suitable functions of $\hat{T}$. The important point to note is that there has been no fundamental change in the solution, but merely a reordering of the error terms according to the new size of the perturbation, with the q_2 and q_3 terms being interchanged as compared to the state for $a < \frac{1}{3}$.

However, the shelf terms do not continue simply to exchange places with the terms further down the perturbation expansion. A major alteration occurs when the $O(F^{1+a})$ and $O(F^{2-a})$ terms are of the same size, at $a = \frac{1}{2}$, where we have

$$\phi' = q_1 + Fq_2 + F^{\frac{3}{2}} q_3 + \cdots. \tag{4.43}$$

Then the $O(F^{\frac{3}{2}})$ terms are given by

$$\mathscr{L} q_3 \equiv -c_1 q_{3\hat{x}} + \beta_0 (q_1 q_3)_{\hat{x}} + \gamma_0 q_{3\hat{x}\hat{x}\hat{x}} = G_3, \tag{4.44}$$

where

$$G_3 = -q_{1\hat{T}} - \delta_0 q_{1\hat{x}\hat{x}} - \phi_0 \alpha_0 [2c_1 q_{1\hat{x}\hat{x}} + (n+1)(q_1^2)_{\hat{x}\hat{x}}] - \phi_0 P_1 [(q_{1\hat{x}})^2 + q_1 q_{1\hat{x}\hat{x}}], \tag{4.45}$$

and q_1 is defined in (4.39). As before, for compatibility we must have

$$\int_{-\infty}^{\infty} G_3 q_1 \, \mathrm{d}\hat{x} = 0, \tag{4.46}$$

but this time the additional terms do affect the secularity condition. Instead of the differential equation for κ given in (4.16), we now obtain

$$\hat{\kappa}_{\hat{T}} = b\hat{\kappa}^3 + d\hat{\kappa}^5, \tag{4.47}$$

where

$$b = \tfrac{8}{15}\delta_0, \quad d = \tfrac{64}{15}\alpha_0 \phi_0 \gamma_0 \left[1 + \frac{12(n+1)}{7\beta_0} + \frac{6P_1}{7\alpha_0 \beta_0}\right]. \tag{4.48}$$

Thus we have an additional term proportional to $\hat{\kappa}^5$ on the right-hand side. Equation (4.47) can be integrated to give

$$\ln(z+1) - z = \frac{2b^2}{d}(\hat{T} - \hat{T}_0), \tag{4.49}$$

where

$$z = b/d\hat{\kappa}^2, \tag{4.50}$$

and $\hat{T}_0$ is a constant. For z large, i.e. $\hat{\kappa}$ small, this reduces to the original dependence (4.17) of κ on T. This rescaling is equivalent to putting $d = 0$ in (4.47). For z small, i.e. $\hat{\kappa}$ large, we can expand the logarithmic term in powers of z to give the approximation

$$-\tfrac{1}{2}z^2 = \frac{2b^2}{d}(\hat{T} - \hat{T}_0), \tag{4.51}$$

so that

$$\hat{\kappa} = \left[\frac{1}{4d(\hat{T}_0 - \hat{T})}\right]^{\frac{1}{4}}, \tag{4.52}$$

which gives a new dependence of the amplitude on the time, from which it can be seen that the singularity is weakened and no longer depends on b.

To formalize the small-z approximation, which is valid for $a > \frac{1}{2}$, we can think of a as giving a scaling for κ, which in turn determines a scaling on T. Thus if we leave the timescale arbitrary for the moment, by writing

$$T - T_0 = F^m T', \tag{4.53}$$

but still use the scalings

$$\phi = \phi_0 + F^{2-2a}\phi', \quad v = F^{2-2a} v', \quad \kappa = F^{-a}\kappa', \quad \hat{x} = F^{-a}(X - x_0 - \xi), \tag{4.54}$$

as before, then the time derivative becomes

$$\frac{\partial}{\partial t} = F^2\frac{\partial}{\partial \tau} + F^{3-m}\frac{\partial}{\partial T'} - F^{-a}\alpha_0\frac{\partial}{\partial \hat{x}} - F^{2-3a}c_1\frac{\partial}{\partial \hat{x}}. \tag{4.55}$$

If we again look for quasi-stationary solutions independent of τ, then instead of (4.29) we now end up with

$$\begin{aligned} -c_1\phi'_{\hat{x}} + \beta_0\phi'\phi'_{\hat{x}} + \gamma_0\phi'_{\hat{x}\hat{x}\hat{x}} = &-F^{2-2a}\left[2c_1\frac{n+1}{\phi_0}\phi'\phi'_{\hat{x}} + \frac{\gamma_0 c_1}{\alpha_0}\phi'_{\hat{x}\hat{x}\hat{x}} + \frac{\gamma_0}{(1-\phi_0)}(\phi'^2)_{\hat{x}\hat{x}\hat{x}}\right] \\ &-F^{2-2a}\frac{(n+1)(n+2)}{2\phi_0^2}[3\phi_0 - 2n(1-\phi_0)]\phi'^2\phi'_{\hat{x}} \\ &-F^{2-a}\phi_0\alpha_0[2c_1\phi'_{\hat{x}\hat{x}} + (n+1)(\phi'^2)_{\hat{x}\hat{x}}] \\ &-F^{2-a}\phi_0 P_1[(\phi'_{\hat{x}})^2 + \phi'\phi'_{\hat{x}\hat{x}}] - F^{1-m+3a}[\phi'_{T'}] - F^{1+a}[\delta_0\phi'_{\hat{x}\hat{x}}] \\ &+O(F^{4-4a}, F^{4-3a}, F^{6-6a}, F^{3-m+2a}, F^{3-m+a}). \end{aligned} \tag{4.56}$$

For a slowly varying soliton solution to exist, the $F^{1-m+3a}\phi'_{T'}$ term (which had the same magnitude as the $O(F^{2-a})$ terms at $a = \frac{1}{2}$) must remain of this order, and so therefore we impose the constraint

$$m = 4a - 1. \tag{4.57}$$

Accordingly, for a range of a such that $a > \frac{1}{2}$, we set

$$T - T_0 = F^{4a-1}T'. \tag{4.58}$$

We now have the asymptotic solution

$$\phi' = q_1 + F^{2-2a}q_2 + F^{2-a}q_3 + F^{1+a}q_4 + F^{4-4a}q_5 + \cdots, \tag{4.59}$$

and the secularity condition is given by

$$\int_{-\infty}^{\infty} G_4 q_1\,\mathrm{d}\hat{x} = 0, \tag{4.60}$$

where

$$G_4 = -q_{1T'} - \phi_0\alpha_0[2c_1 q_{1\hat{x}\hat{x}} + (n+1)(q_1^2)_{\hat{x}\hat{x}}] - \phi_0 P_1[(q_{1\hat{x}})^2 + q_1 q_{1\hat{x}\hat{x}}]. \tag{4.61}$$

This yields the following differential equation for $\hat{\kappa}$ as a function of T':

$$\hat{\kappa}_{T'} = d\hat{\kappa}^5, \tag{4.62}$$

with d as in (4.48). Integrating (4.62) gives

$$\hat{\kappa} = \left[\frac{1}{4d(T'_0 - T')}\right]^{\frac{1}{4}}, \tag{4.63}$$

which agrees with the small-z approximation of (4.49). Therefore, the two regions governed by the different dependences of $\hat{\kappa}$ on the time in (4.40) and (4.63) are matched via the intermediate region given by (4.49). The displacement ξ can also be found using (4.28), (4.58) and (4.63), and can be written

$$\xi = -F^{2a-2}\frac{4\gamma_0}{d^{\frac{1}{2}}}(T'_0 - T')^{\frac{1}{2}}. \tag{4.64}$$

Thus, it becomes $O(1)$ when $a = 1$, and the original logarithmic dependence on the time, given by (4.20), has now been replaced by a square-root dependence.

We suggest that there are in fact no further changes in the dependence of $\hat{\kappa}$ on T' until $a = 1$, when there is a major alteration in the solution.

5. Finite-amplitude solitary waves

In this section, we examine what happens at $a = 1$, when there is a fundamental change in the solution. Then, the soliton amplitude has become $O(1)$, the soliton velocity c_1 in the moving frame has become of the same order as the velocity of the translating frame itself, and all the nonlinear terms of the form F^{2n-2na} in the perturbation expansion for ϕ' have come together and are $O(1)$ too. Now the equations are fully nonlinear, and we demonstrate that they have solitary wave solutions for the voidage fraction at leading order. These $O(1)$ solitary waves cannot be expressed in closed form and the equation for their profile must be integrated numerically, but information about the conditions necessary for their existence can be obtained from consideration of the phase-plane diagram. An integral expression can be found, relating the velocity to the amplitude of the wave, which is a common feature in nonlinear wave theory.

If we retain the perturbation terms that are $O(F)$ compared to the leading-order solitary wave terms, then we can apply a multiple-scales analysis in the same way as for the KdV soliton, and allow the velocity (and hence amplitude) to vary on a long timescale. An integral form can be found for the secularity condition, but again an explicit expression is unavailable. However, if we consider the amplitude of the solitary wave to be small, then it can be shown that this secularity condition reduces to the one in (4.60) and (4.61) for $\hat{\kappa}$ when $a > \frac{1}{2}$. This substantiates our claim in the previous section that all the intermediate regions have been found and that the solitary wave does indeed match back onto the KdV soliton. Thus, there is no need for a fast-time-dependent transitional zone and this is an important feature of the analysis. It will also be shown that although the solitary waves are growing initially, they eventually tend to a limiting voidage which only depends on n and ϕ_0.

To find the relevant equations for the fully nonlinear waves, we must return to the original system (2.12) and (2.13). Then, we rescale the variables using

$$\phi = \hat{\phi}, \quad v = \hat{v}, \quad \zeta = F^{-1}(x - x_0 - \hat{\xi}), \quad T - T_0 = F^2\hat{T}, \tag{5.1}$$

where $\hat{\phi}$, $\hat{v}$, ζ and $\hat{T}$ are $O(1)$, and write

$$\partial\hat{\xi}/\partial\hat{T} = F^{-1}c. \tag{5.2}$$

Substituting these into (2.12) and (2.13), we obtain

$$c\hat{\phi}_\zeta + [(1-\hat{\phi})\,\hat{v}]_\zeta = 0, \tag{5.3}$$

$$(1-\hat{\phi})\left(\frac{\phi_0}{\hat{\phi}}\right)^{n+1}\left(1-\frac{\hat{v}}{\phi_0}\right) - (1-\hat{\phi}) + \frac{1}{R}\hat{v}_{\zeta\zeta} = F[(1-\hat{\phi})(\hat{v}-c)\,\hat{v}_\zeta + p_s'(\hat{\phi})\,\hat{\phi}_\zeta], \tag{5.4}$$

where the neglected terms are $O(F^2)$ and c is now an as yet arbitrary velocity.

If we use a perturbation expansion for $\hat{\phi}$, $\hat{v}$ in powers of F,

$$\hat{\phi} = \hat{\phi}_0 + F\hat{\phi}_1 + \cdots, \quad \hat{v} = \hat{v}_0 + F\hat{v}_1 + \cdots, \tag{5.5}$$

then the new, *fully nonlinear* equations have leading-order solutions $\hat{\phi}_0, \hat{v}_0$ satisfying the simultaneous equations

$$c\hat{\phi}_{0\zeta} + [(1-\hat{\phi}_0)\,\hat{v}_0]_\zeta = 0, \tag{5.6}$$

$$(1-\hat{\phi}_0)\left(\frac{\phi_0}{\hat{\phi}_0}\right)^{n+1}\left(1-\frac{\hat{v}_0}{\phi_0}\right) - (1-\hat{\phi}_0) + \frac{1}{R}\hat{v}_{0\zeta\zeta} = 0. \tag{5.7}$$

Assuming that the bed is undisturbed far above, so that $\hat{\phi}_0 \to \phi_0$ and $\hat{v}_0 \to 0$ as $\zeta \to \infty$, the first of these can be integrated to give

$$\hat{v}_0 = -c\frac{\hat{\phi}_0 - \phi_0}{1-\hat{\phi}_0}. \tag{5.8}$$

This expression for $\hat{v}_0$ can be used to eliminate the particle velocity from the momentum equation to yield an equation for the voidage alone. Substituting (5.8) into (5.7) and temporarily dropping the zero subscripts on $\hat{\phi}$ gives

$$\frac{c(1-\phi_0)}{R}\frac{\partial}{\partial\zeta}\left[\frac{\hat{\phi}_\zeta}{(1-\hat{\phi})^2}\right] = -(1-\hat{\phi})+(1-\hat{\phi})\left(\frac{\phi_0}{\hat{\phi}}\right)^{n+1}\left(1+\frac{c}{\phi_0}\frac{\hat{\phi}-\phi_0}{1-\hat{\phi}}\right). \tag{5.9}$$

Despite its formidable nonlinearity, equation (5.9) can in fact be integrated once, to yield

$$\frac{c(1-\phi_0)}{2R}\frac{\hat{\phi}_\zeta^2}{(1-\hat{\phi})^4} = h(\hat{\phi}), \tag{5.10}$$

where
$$h(\hat{\phi}) = \ln\left(\frac{1-\hat{\phi}}{1-\phi_0}\right)+\phi_0^n\int_{\phi_0}^{\hat{\phi}}\left[\frac{\phi_0-c}{z^{n+1}(1-z)}+\frac{c(1-\phi_0)}{z^{n+1}(1-z)^2}\right]\mathrm{d}z. \tag{5.11}$$

Obviously, $h(\phi_0) = 0$, and this corresponds to the uniform state far above the disturbance, but we also require $h(\hat{\phi}) > 0$ for a range of $\hat{\phi} > \phi_0$ if solutions of (5.10) are to exist. If we consider

$$h'(\hat{\phi}) = -\frac{1}{1-\hat{\phi}}+\frac{1}{1-\hat{\phi}}\left(\frac{\phi_0}{\hat{\phi}}\right)^{n+1}\left(1+\frac{c}{\phi_0}\frac{\hat{\phi}-\phi_0}{1-\hat{\phi}}\right), \tag{5.12}$$

it is easy to see that $h'(\phi_0) = 0$, and we can also show that

$$h''(\phi_0) = \frac{1}{\phi_0(1-\phi_0)^2}(c-\alpha_0), \tag{5.13}$$

which is positive if $c > \alpha_0$. Hence $h(\hat{\phi})$ has a minimum at $\phi = \phi_0$, lying on the axis. We also note that $h(\hat{\phi}) \to +\infty$ as $\hat{\phi} \to 1$. If, in addition, we have $h(A) = 0$, where A is a constant satisfying $\phi_0 < A < 1$, then we will have a closed path in the phase plane on taking the square root of (5.10). This represents a periodic solution or, if the period is infinite, we have a homoclinic orbit and thus a solitary wave solution to the strongly nonlinear equations. Needham & Merkin (1986) investigate periodic travelling waves of the full equations using bifurcation theory and numerical integrations. Here, however, we have solitary wave solutions as we are in the long-wavelength limit.

For an appropriate zero of $h(\hat{\phi})$ at $\hat{\phi} = A$, we must have at least two turning points of $h(\hat{\phi})$ in the range $(\phi_0, 1)$ and so we consider what happens when $h'(\hat{\phi}) = 0$. Putting the right-hand side of (5.12) equal to zero and rearranging as an equation for c in terms of $\hat{\phi}$ gives

$$c = \phi_0\frac{1-\hat{\phi}}{\hat{\phi}-\phi_0}\left[\left(\frac{\hat{\phi}}{\phi_0}\right)^{n+1}-1\right]. \tag{5.14}$$

This function has a single maximum at $\hat{\phi} = \phi_c$, where

$$n/(n+2) < \phi_c < 1, \tag{5.15}$$

provided
$$n/(n+2) > \phi_0. \tag{5.16}$$

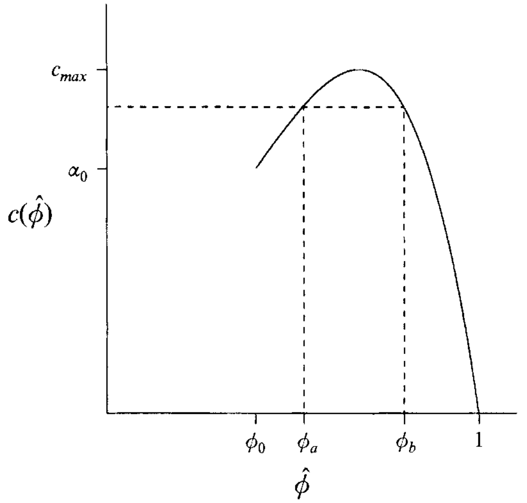

FIGURE 1. A plot of solitary wave speed c as a function of $\hat{\phi}$ for $h'(\hat{\phi}) = 0$.

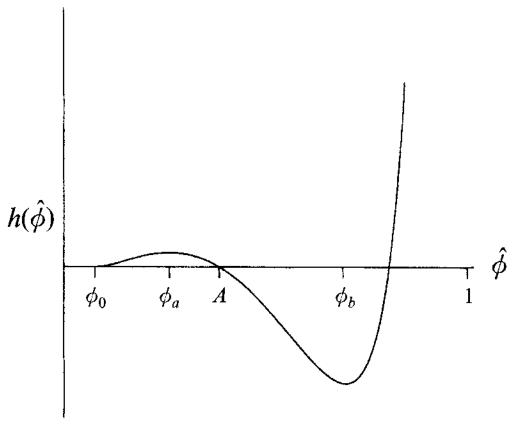

FIGURE 2. The graph of $h(\hat{\phi})$ for $n = 3$, $\phi_0 = 0.4$ and $A = 0.6$.

This restriction on ϕ_0 is *precisely* the condition we imposed in (3.22) in order to have $\beta_0 > 0$ and therefore solitons of high voidage moving through the bed. It is certainly significant that it occurs again here as a *necessary* condition for the existence of high-voidage, $O(1)$ solitary waves.

A plot of c against $\hat{\phi}$ for $h'(\hat{\phi}) = 0$ is illustrated in figure 1 for $\phi_0 = 0.4$ and $n = 3$. Examining the graph, we can see that there exist exactly two values ϕ_a, ϕ_b such that $h'(\phi_a) = 0 = h'(\phi_b)$ and $\phi_0 < \phi_a, \phi_b < 1$ provided that

$$\alpha_0 < c < c_{max}. \tag{5.17}$$

Thus for c lying in this range, we have a maximum and a minimum of $h(\hat{\phi})$ for $\phi_0 < \hat{\phi} < 1$, but we also require the additional condition $h(\phi_b) \leqslant 0$ for there to be a zero of $h(\hat{\phi})$ in the interval $(\phi_a, \phi_b]$. It can be proved that there exists a range of values of ϕ_b for which this holds, by using a continuity argument. Noting that when $\phi_a = \phi_0$, $c = \alpha_0$, and ϕ_0 is then a stationary point of inflexion of $h(\hat{\phi})$, so it follows that $h(\phi_b)$ is necessarily less than zero. As c and h are continuous functions, there must be a range of values ϕ_b such that $h(\phi_b) < 0$. A graph of $h(\phi)$ against ϕ is shown in figure 2, where A is the amplitude of the solitary wave. The maximum $\hat{\phi}$ for possible solitary wave solutions has $h(\phi_b) = 0$ so that the turning point at $\hat{\phi} = \phi_b$ is on the axis. Then $h(\hat{\phi}) > 0$ for $\phi_0 < \hat{\phi} < \phi_b$.

A paper by Barcilon & Richter (1986) which is concerned with magma migration through a solid permeable matrix examines a set of equations which resemble those under study here (see Appendix B). Following their analysis, it is helpful to define

$$\Gamma \equiv \Gamma(\hat{\phi};\phi_0,n)$$
$$= -\left\{\ln\left(\frac{1-\hat{\phi}}{1-\phi_0}\right) + \int_{\phi_0}^{\hat{\phi}} \frac{\phi_0^{n+1}}{z^{n+1}(1-z)}\,\mathrm{d}z\right\}\left\{\int_{\phi_0}^{\hat{\phi}} \frac{\phi_0^n(z-\phi_0)}{z^{n+1}(1-z)^2}\,\mathrm{d}z\right\}^{-1} \tag{5.18}$$

and to write (5.10) in the form

$$\frac{c(1-\phi_0)}{2R}\frac{\hat{\phi}_\xi^2}{(1-\hat{\phi})^4} = (c-\Gamma(\hat{\phi};\phi_0,n))\left\{\int_{\phi_0}^{\hat{\phi}} \frac{\phi_0^n(z-\phi_0)}{z^{n+1}(1-z)^2}\,\mathrm{d}z\right\}. \tag{5.19}$$

Then if $h(\hat{\phi}) = 0$ at the amplitude $\hat{\phi} = A$, we must have

$$c = \Gamma(A;\phi_0,n), \tag{5.20}$$

as the integral on the right-hand side of (5.19) is positive for $\phi_0 < A < 1$. Equation (5.20) exhibits the dependence of the solitary wave velocity on the amplitude A of the wave.

Nakayama & Mason (1992) examine a set of equations for magma migration which are very similar to those used by Barcilon & Richter (1986), and they have developed a different method for determining the conditions under which solitary wave solutions can exist. In fact, their method gives sufficient conditions, whereas ours yields necessary ones. We will apply their method to the fluidized-bed equations by writing (5.19) as

$$\hat{\phi}_\xi^2 = \frac{2R}{1-\phi_0}(1-\hat{\phi})^4\frac{(F(A)-F(\hat{\phi}))}{F(A)}\left\{\int_{\phi_0}^{\hat{\phi}} \frac{\phi_0^n(z-\phi_0)}{z^{n+1}(1-z)^2}\,\mathrm{d}z\right\}, \tag{5.21}$$

where
$$F(\hat{\phi}) = \Gamma(\hat{\phi};\phi_0,n). \tag{5.22}$$

Differentiating $F(\hat{\phi})$ with respect to $\hat{\phi}$ yields

$$\frac{\mathrm{d}F}{\mathrm{d}\hat{\phi}} = \frac{\phi_0^n}{\hat{\phi}^{n+1}(1-\hat{\phi})^2}\left\{\int_{\phi_0}^{\hat{\phi}} \frac{G(z;\hat{\phi})}{z^{n+1}(1-z)^2}\,\mathrm{d}z\right\}\left\{\int_{\phi_0}^{\hat{\phi}} \frac{\phi_0^n(z-\phi_0)}{z^{n+1}(1-z)^2}\,\mathrm{d}z\right\}^{-2}, \tag{5.23}$$

where

$$G(z;\hat{\phi}) = (\hat{\phi}^{n+1}-\phi_0^{n+1})(1-\hat{\phi})(z-\phi_0)-(1-z)(\hat{\phi}-\phi_0)(z^{n+1}-\phi_0^{n+1}). \tag{5.24}$$

We note that

$$G(\phi_0;\hat{\phi}) = 0 = G(\hat{\phi};\hat{\phi}) \tag{5.25}$$

and
$$\mathrm{d}^2G/\mathrm{d}z^2 = -(n+1)(n-(n+2)z)z^{n-1}(\hat{\phi}-\phi_0). \tag{5.26}$$

Thus for $\hat{\phi} > \phi_0$ and $n/(n+2) > z$, $\mathrm{d}^2G/\mathrm{d}z^2$ is negative, so given that (5.25) holds, we must have $G(z;\hat{\phi}) > 0$ for $\phi_0 < z < \hat{\phi}$. Thus

$$\mathrm{d}F/\mathrm{d}\hat{\phi} > 0, \tag{5.27}$$

for $n/(n+2) > \hat{\phi} > \phi_0$, and so F is an increasing function of $\hat{\phi}$ in this range. Therefore

$$F(A) > F(\hat{\phi}) \tag{5.28}$$

if
$$n/(n+2) > A > \hat{\phi} > \phi_0. \tag{5.29}$$

It can easily be shown that $F(\hat{\phi}) > 0$ for $1 > \hat{\phi} > \phi_0$ and so we have $\hat{\phi}_\xi^2 > 0$ from (5.21)

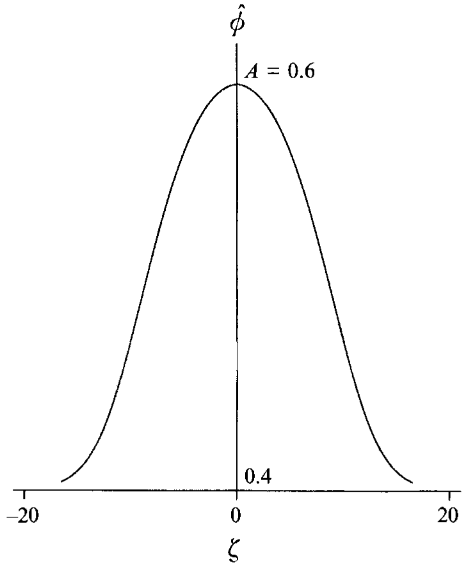

FIGURE 3. A numerically integrated solitary wave profile for $n = 3$, $\phi_0 = 0.4$ and $A = 0.6$. For the typical case of glass beads of 50 μm diameter, the solitary wave is approximately 0.5 cm wide and has a velocity of the order of 5.5 cm s^{-1}.

for $\hat{\phi}$ and A satisfying these restrictions. However, although condition (5.29) is sufficient to ensure solitary wave solutions, it is not necessary and there is, in fact, a larger range of $\hat{\phi}$ for which $\hat{\phi}_\zeta^2 > 0$.

The profile given by (5.19) can be integrated numerically, to exhibit the shape of the solitary wave, using Simpson's rule and a fourth-order Runge–Kutta method, with the initial conditions specified at a distance ϵ away from A (Barcilon & Richter 1986; Harris 1992). An example of the type of profile obtained is shown in figure 3, with $n = 3$ and $\phi_0 = 0.4$.

In order to match the solitary wave profile back onto the growing KdV soliton, we must allow the velocity c (and hence the amplitude A) to vary on a timescale T', where $T - T_0 = F^3 T'$. This results in an extra term $F\hat{\phi}_{T'}$ on the right-hand side of (5.3), so that we now have

$$c\hat{\phi}_\zeta + [(1-\hat{\phi})\hat{v}]_\zeta = F\hat{\phi}_{T'}. \tag{5.30}$$

Using (5.30), (5.4) and the perturbation expansion given in (5.5), we consider the $\hat{\phi}_1$ and $\hat{v}_1$ terms. It is convenient to write these equations in the matrix form

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \hat{\phi}_1 \\ \hat{v}_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{5.31}$$

where

$$\left.\begin{aligned} p &= c(1-\hat{\phi}_0)\frac{\partial}{\partial\zeta}\left(\frac{1}{1-\hat{f}_0}\right), \quad q = \frac{\partial}{\partial\zeta}(1-\hat{\phi}_0), \\ r &= \left[1-\left(\frac{\phi_0}{\hat{\phi}_0}\right)^{n+1}\left(1-\frac{\hat{v}_0}{\phi_0}\right)\left(1+\frac{(n+1)(1-\hat{\phi}_0)}{\hat{\phi}_0}\right)\right], \\ \nu &= \frac{1-\hat{\phi}_0}{\phi_0}\left(\frac{\phi_0}{\hat{\phi}_0}\right)^{n+1}, \quad s = -\nu + \frac{1}{R}\frac{\partial^2}{\partial\zeta^2}, \\ a_1 &= \hat{\phi}_{0T'}, \quad a_2 = (1-\hat{\phi}_0)(\hat{v}_0 - c)\hat{v}_{0\zeta} + p_s'(\hat{\phi}_0)\hat{\phi}_{0\zeta}. \end{aligned}\right\} \tag{5.32}$$

The adjoint homogeneous problem is then given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.33}$$

where

$$\left.\begin{aligned} P &= -c\frac{1-\phi_0}{1-\hat{\phi}_0}\frac{\partial}{\partial\zeta}, \quad Q = r, \\ R &= -(1-\hat{\phi}_0)\frac{\partial}{\partial\zeta}, \quad S = -\nu+\frac{1}{R}\frac{\partial^2}{\partial\zeta^2}. \end{aligned}\right\} \tag{5.34}$$

Then, if $(l_1, l_2)^T$ satisfies (5.33), the secularity condition is given by

$$0 = \int_{-\infty}^{\infty} (l_1 a_1 + l_2 a_2)\,\mathrm{d}\zeta. \tag{5.35}$$

It can be verified that a solution to the homogeneous adjoint problem is given by

$$l_{1\zeta} = -\frac{r\hat{\phi}_{0\zeta}}{1-\hat{\phi}_0}, \quad l_2 = \hat{v}_{0\zeta}, \tag{5.36}$$

and so this can be used to evaluate the integral given in (5.35).

In order to demonstrate that all the changes in the secularity condition have indeed been found for the whole time interval before the equations become strongly nonlinear at $a = 1$, we examine the solitary wave secularity condition, setting

$$\left.\begin{aligned} &\hat{\phi}_0 \to \phi_0 + F^{2-2a}\phi_1 + \cdots, \\ &\hat{v}_0 \to F^{2-2a}v_1 + \cdots, \\ &c \to \alpha_0 + F^{2-2a}c_1, \end{aligned}\right\} \tag{5.37}$$

$$\frac{\partial}{\partial\zeta} \to F^{1-a}\frac{\partial}{\partial\hat{x}}, \quad \frac{\partial}{\partial T'} \to F^{3-m}\frac{\partial}{\partial T'}. \tag{5.38}$$

First of all, we must integrate the equation for $l_{1\hat{x}}$, to obtain

$$l_1 = F^{2-2a}(n+1)\frac{\phi_1}{\phi_0} + O(F^{4-4a}). \tag{5.39}$$

Then, substituting into the secularity condition (5.35) and just retaining the leading-order terms, we obtain

$$\int_{-\infty}^{\infty} F^{6-m-3a}(n+1)\frac{\phi_1}{\phi_0}\phi_{1T'}\,\mathrm{d}\hat{x} - \int_{-\infty}^{\infty} F^{7-7a}\,2\alpha_0(n+1)\left[c_1+(n+1)\phi_1+\frac{P_1\phi_1}{2\alpha_0}\right]\phi_{1\hat{x}}^2\,\mathrm{d}\hat{x} = 0. \tag{5.40}$$

But $F^{6-m-3a} = F^{7-7a}$ if $m = 4a-1$, which is exactly the scaling for $a > \frac{1}{2}$, obtained in (4.57). Therefore, we have

$$\int_{-\infty}^{\infty} \left\{\phi_1\phi_{1T'} - 2\alpha_0\phi_0\left[c_1+(n+1)\phi_1+\frac{P_1\phi_1}{2\alpha_0}\right]\phi_{1\hat{x}}^2\right\}\mathrm{d}\hat{x} = 0. \tag{5.41}$$

This is the same as the secularity condition (4.60), (4.61) for $a > \frac{1}{2}$ when we put $q_1 = \phi_1$ and integrate by parts once. This confirms the earlier claim that there were no further changes in the dependence of $\hat{\kappa}$ on the time after (4.63), before the soliton becomes an $O(1)$ solitary wave. Further, if we substitute (5.37) and (5.38) back into (5.10) and (5.11), then we can also recover the 'sech-squared' solution for ϕ_1, and we conclude that the fully nonlinear solitary wave matches directly back onto the amplifying soliton, without the need for a non-adiabatically varying time-dependent region.

We still need to investigate the amplitude variation to find out what happens to the solitary wave. In Appendix B, we consider the special case $n = 2$, $\phi_0 \ll 1$ which is appropriate for magma flows (Barcilon & Richter 1986). This provides insight into the behaviour, as explicit solutions can be found, but an artefact of the scalings used means that the amplitude continues to increase, with no finite limit. In the fluidized bed context, there is an upper limit on the amplitude and to see this we first rewrite the secularity condition (5.35) in the form

$$\mathrm{d}I/\mathrm{d}T' = J, \tag{5.42}$$

where
$$J = -\int_{-\infty}^{\infty} l_2 a_2 \,\mathrm{d}\zeta$$

$$= \int_{-\infty}^{\infty} c(1-\phi_0)\left[c^2\frac{(1-\phi_0)^2}{(1-\hat{\phi}_0)^2} + p_s'(\hat{\phi}_0)\right]\frac{\hat{\phi}_{0\zeta}^2}{(1-\hat{\phi}_0)^2}\,\mathrm{d}\zeta, \tag{5.43}$$

and
$$\frac{\mathrm{d}I}{\mathrm{d}T'} = \int_{-\infty}^{\infty} l_1 a_1 \,\mathrm{d}\zeta. \tag{5.44}$$

The integrand in (5.43) is positive for $A > \hat{\phi}_0 > \phi_0$, so that $\mathrm{d}I/\mathrm{d}T' > 0$. As in the case in Appendix B, I is a function of A, $G(A)$ say, and so

$$\frac{\mathrm{d}I}{\mathrm{d}T'} = G'(A)\frac{\mathrm{d}A}{\mathrm{d}T'} > 0. \tag{5.45}$$

Initially $\mathrm{d}A/\mathrm{d}T' > 0$, as the secularity condition then reduces to that of the amplifying soliton. Thus $G'(A)$ is positive for a range of $A > \phi_0$, as G' is a continuous function of A. But G' cannot change sign, because of (5.45), and so it remains positive. Thus $\mathrm{d}A/\mathrm{d}T'$ also remains positive and therefore the soliton amplitude is a monotonically increasing function of time.

We now show that A has a limiting value and tends to this exponentially. The only feasible candidate is the maximum voidage for the existence of solitary wave solutions of (5.10) and (5.11), which is $\phi = \phi_b$, where $h(\phi_b) = h'(\phi_b) = 0$. Thus, the turning point ϕ_b of h lies on the axis, so $h(\phi) > 0$ for $\phi_0 < \phi < \phi_b$. Equations (5.30) and (5.4) are still appropriate, as ϕ, v, c and the wave amplitude A remain $O(1)$, and perturbing the voidage and particle velocity around $\phi = \phi_b$ we have

$$\phi_b - \phi = F\phi_1 + F^2\phi_2 + \cdots, \quad v = v_0 + Fv_1 + F^2v_2 + \cdots. \tag{5.46}$$

The wave velocity c can also be expanded in powers of F using a Taylor series for Γ, but the condition $h'(\phi_b) = 0$ implies that $\Gamma'(\phi_b; \phi_0, n) = 0$ too. Therefore there is no $O(F)$ term, so

$$c = c_0 + F^2c_2 + \cdots, \tag{5.47}$$

and
$$c_0 = \phi_0\frac{1-\phi_b}{\phi_b-\phi_0}\left[\left(\frac{\phi_b}{\phi_0}\right)^{n+1} - 1\right], \tag{5.48}$$

from (5.14). The appropriate leading-order solution for v can be found by insisting on matching to $\hat{v}_0$ in (5.8), and is given by

$$v_0 = \phi_0\left[1-\left(\frac{\phi_b}{\phi_0}\right)^{n+1}\right]. \tag{5.49}$$

Substituting the perturbation expansions (5.46) and (5.47) into (5.30) and (5.4) shows that ϕ_1 and v_1 satisfy

$$v_1 = c_0\frac{1-\phi_0}{(1-\phi_b)^2}\phi_1, \tag{5.50}$$

$$c_0\frac{1-\phi_0}{R}\frac{\phi_{1\zeta}^2}{(1-\phi_b)^4} = \phi_1^2 h''(\phi_b). \tag{5.51}$$

But ϕ_b is a minimum turning point of h, and so therefore $h''(\phi_b) > 0$ and we can write

$$\phi_{1\zeta\zeta} = \Omega^2\phi_1, \tag{5.52}$$

where

$$\Omega^2 = \frac{R(1-\phi_b)^2}{c_0(1-\phi_0)}\left[\frac{1-\phi_0}{\phi_b-\phi_0}\left[1-\left(\frac{\phi_0}{\phi_b}\right)^{n+1}\right]-\frac{n+1}{\phi_b}(1-\phi_b)\right]. \tag{5.53}$$

Therefore

$$\phi_1(\zeta, T') = B(T')\cosh\Omega\zeta, \tag{5.54}$$

as we require ϕ_1 to be symmetrical about $\zeta = 0$. To find B as a function of T', we need to examine higher-order terms.

The second-order solutions ϕ_2 and v_2 satisfy

$$-c_0\frac{1-\phi_0}{1-\phi_b}\phi_{2\zeta}+(1-\phi_b)v_{2\zeta} = -\phi_{1T'}-2c_0\frac{1-\phi_0}{(1-\phi_b)^2}\phi_1\phi_{1\zeta}, \tag{5.55}$$

$$\frac{n+1}{\phi_b}(1-\phi_b)\phi_2-\frac{v_2}{\phi_0}(1-\phi_b)\left(\frac{\phi_0}{\phi_b}\right)^{n+1}+\frac{v_{2\zeta\zeta}}{R} = -\left[p_s'(\phi_b)+c_0^2\frac{(1-\phi_0)^2}{(1-\phi_b)^2}\right]\phi_{1\zeta}+\sigma\phi_1^2, \tag{5.56}$$

where

$$\sigma = (n+1)(n+2)\frac{1-\phi_b}{2\phi_b^2}-\frac{n+1}{\phi_b}+\frac{c_0}{\phi_0}\frac{1-\phi_0}{(1-\phi_b)^2}\left(\frac{\phi_0}{\phi_b}\right)^{n+1}+\frac{c_0}{\phi_0}\frac{1-\phi_0}{1-\phi_b}\frac{n+1}{\phi_b}\left(\frac{\phi_0}{\phi_b}\right)^{n+1}. \tag{5.57}$$

It can be shown that for a solution to exist, we must have

$$-\left[p_s'(\phi_b)+c_0^2\frac{(1-\phi_0)^2}{(1-\phi_b)^2}\right]B\Omega^2 = \frac{n+1}{c_0\phi_b}\frac{(1-\phi_b)^2}{1-\phi_0}\frac{dB}{dT'}. \tag{5.58}$$

Therefore

$$dB/dT' = -lB, \tag{5.59}$$

where $l > 0$, and thus we have

$$B(T') = B_0 e^{-lT'}, \tag{5.60}$$

with B_0 a positive constant. From (5.54) and (5.46),

$$\phi = \phi_b - FB_0 e^{-lT'}\cosh\Omega\zeta, \tag{5.61}$$

which describes the peak of the solitary wave near $\phi = \phi_b$, and the amplitude A is given by

$$A = \phi_b - FB_0 e^{-lT'}, \tag{5.62}$$

so that as $T' \to \infty$, $A \to \phi_b$. Therefore ϕ_b is the limiting size of the solitary wave and is reached exponentially. We also note that when the solitary wave was first formed, T' was negative, with $T_0 - T = -F^3 T'$, so that as T' passes through zero, T becomes larger than T_0 and thus we no longer have a finite-time singularity.

6. Comparison with experimental data

In this section, we give numerical estimates for the important quantities and parameters in the theory, to see whether these are in agreement with various experimental data and observations.

First, we note that if (3.22) holds, then $\alpha_0 > 1$ from (3.3), so the dimensional linear wave velocity $\alpha_0 U_0$ is greater than the interstitial fluid velocity U_0. Thus, infinitesimal voidage waves propagate through the bed at a higher speed than the fluid. In fact, Geldart (1973) suggests classifying beds of solid particles into four groups, characterized by the density difference between the particles and the fluidizing gas and the mean particle size. Two of these groups, A and B, have bubbles rising more rapidly than the interstitial gas (Couderc 1985). The remaining two correspond to cohesive powders, which are difficult to fluidize, and powders of large or very dense particles which tend to spout (where the fluid only takes a single path up through the bed) and where all but the largest bubbles rise more slowly than the interstitial velocity. Therefore, our model seems to be appropriate for describing the fluidization of powders in groups A and B. The difference between the two is that group A powders exhibit uniform expansion before they bubble and a maximum bubble size seems to exist, whereas group B particles bubble immediately upon fluidization and bubble size appears to be independent of particle size.

In the linear perturbation analysis given in §3, the short-timescale linear wave equation (3.4) and the longer timescale equation (3.5) were derived from (3.2) by using the approximations

$$F^2 \ll 1, \quad F^2/R \ll 1. \tag{6.1}$$

In fact, the stronger condition $F \ll 1$ is also required for our analysis as, in the perturbed KdV equation (4.1), the perturbation terms are $O(F)$ and must be small to ensure the slowly varying soliton solution is valid.

As indicated previously, we can make estimates of these quantities using experimental data by Schügerl (1971) if we take $\nu_s = \mu_s/\rho_s$ to be the measured kinematic bed viscosity. However, we note that the velocities given in the paper are superficial and not interstitial gas velocities, so that we must divide the given value by $\phi_0 = 0.4$ for use in our theory.

For glass beads of 50 μm diameter, typical values of the velocity and kinematic viscosity are approximately 5.7 cm s^{-1} and 1.9 cm^2 s^{-1} respectively. In order to have the ratio F^2/R less than unity, we require the disturbance lengthscale h to exceed 0.1 cm, which is approximately 20 particle diameters. Thus any h bigger than this will satisfy the continuum hypothesis requirement that the lengthscales be large compared to the particle size and spacing. For an example, we take $h = 1$ cm as was apparently the case in the experiments of El-Kaissy & Homsy (1976), so that $F^2 = 0.03$ and $R = 3$ which agree with our original assumptions of $F^2 \ll 1$ and $R = O(1)$. The superficial wave velocity which should be observable is then 5.5 cm s^{-1}, with a solitary wave width of 0.5 cm. The results are similar when we consider larger but lighter polystyrene spheres of diameter 250 μm, yielding $F^2 = 0.15$ and $R = 6.25$ for the same value of h. The minimum interstitial fluidization velocity for heavier quartz particles of 175 μm and silicon carbide particles of 190 μm diameter is around 25 cm s^{-1}, with a kinematic viscosity of about 10 cm^2 s^{-1}. Thus for $F^2/R < 1$ to hold, we now require a larger value of the voidage disturbance length, i.e. $h > 0.5$ cm. If we use $h = 1$ cm as before, then we have $F^2 = 0.625$ and $R = 2.5$ so that the Froude number is somewhat bigger.

We now examine the various different timescales in the problem. The KdV soliton voidage profile is a valid description at $\tau = O(1)$, or equivalently, when the non-dimensional time $\tilde{t} = O(F^{-2})$, whereas the solitary wave is appropriate when $T = O(1)$ and is $O(F^2)$ away from the finite-time singularity. In terms of dimensional parameters, for the glass beads the KdV solitons will develop in approximately 65 cm, with the fully developed solitary waves forming in 4 m. The corresponding distances for the polystyrene spheres are 15 cm and 35 cm, while those for the quartz and silicon carbide particles are 3 cm and 4 cm.

The estimates given here seem to agree with the accepted experimental observations that beds consisting of smaller (e.g. 50 μm glass beads) or lighter particles (e.g. polystyrene spheres) are more stable than those formed from larger denser particles (e.g. quartz or silicon carbide), as for these former cases any small disturbances will not grow significantly during the upward motion through the bed. For the quartz and silicon carbide particles, however, it can be seen that initially small perturbations grow rapidly and become $O(1)$ within a short distance.

El-Kaissy & Homsy (1976) have made measurements of instability waves in a two-dimensional liquid-fluidized bed for flow rates slightly above that necessary for fluidization. They note that horizontal bands about 1 cm wide of high voidage propagate upward. These coherent, horizontal waves persist at low flow rates. However, if the gas velocity is increased, a small planar region exists near the base of the bed, but higher up the waves become convex and coalesce. If the amplitude is small, then another coherent wave is formed, but if the amplitude is large enough the wave breaks up into high-voidage regions, which are suggestive of bubbles.

The development of transverse structure to such waves has also been observed by Didwania & Homsy (1981) in similar experiments with liquid-fluidized beds which will bubble at a high enough flow rate. However, we note that in liquid-fluidized beds the development is first of a wavy structure, then turbulent and finally bubbling as the fluid velocity is increased, whereas with gas we have bubbling before turbulence. In gas-fluidized beds which expand uniformly before bubbling, we expect similar instability waves to exist and have the same qualitative behaviour. Needham & Merkin (1984) examine cross-channel modes in a two-dimensional model and find that the number of unstable modes increases with increasing bed width, creating transverse structure. However, in our one-dimensional model, we expect coherent planar waves to exist and be stable at all flow rates since there are no such cross-channel modes. The planar waves seem to be described well by our $O(1)$ solitary wave solutions but we expect that these may be unstable to two-dimensional perturbations, as found for the similar magma equations by Barcilon & Lovera (1989).

7. Conclusions

The set of equations governing the behaviour of a uniformly fluidized bed subjected to a disturbance of infinitesimally small amplitude can be reduced to an equation for the voidage perturbation ϕ'. Over short timescales, any such disturbance will simply propagate upwards at a constant velocity and without change in shape, as the behaviour is governed by the standard linear wave equation. On a longer timescale, however, unstable linear amplification takes place, following which nonlinear effects become important and must be included. A balance can then be obtained between these (weak) nonlinear terms and weak dispersive effects. This yields the Korteweg–de Vries equation with either an amplifying or dissipative Burgers perturbation term, depending on the sign of the coefficient. Thus we have

$$\frac{\partial \phi'}{\partial \tau}+\beta_0 \phi' \frac{\partial \phi'}{\partial X}+\gamma_0 \frac{\partial^3 \phi'}{\partial X^3} = -F\delta_0 \frac{\partial^2 \phi'}{\partial X^2}+O(F^2), \tag{7.1}$$

where β_0, γ_0 and δ_0 are constants, τ is the long timescale and X is the spatial coordinate in a translating frame. An additional fourth-derivative is needed to stabilize short-scale disturbances (see (A 18)).

We restrict ourselves to examining the behaviour when $\beta_0 > 0$, which holds when $\phi_0 < n/(n+2)$, where ϕ_0 is the voidage fraction of the uniformly fluidized bed and n is the index in the Richardson–Zaki relation. Experimental measurements of ϕ_0 and n indicate that this criterion for ϕ_0 corresponds to the physically plausible situation for fluidized beds. As we are interested in the formation of bubbles and slugs from voidage inhomogeneities in the bed, we only consider the case $\delta_0 > 0$, when the KdV equation is initially valid, but with small amplification rather than dissipation. As a convenient starting point we consider an initial condition of a KdV soliton. This is permissible, because the KdV equation itself is completely integrable, and so any initial disturbance breaks up quickly into a set of independent solitons, any one of which can then serve as the initial condition for the subsequent development in which the new amplifying or dissipative mechanism comes into play. The KdV soliton represents a wave with a *higher voidage* fraction than the uniform bed, propagating upwards at constant velocity.

We predict the ubiquitous presence of amplifying solitonic pulses of voidage propagating upwards through the bed, while regions with a lower voidage than the uniform bed, i.e. regions of higher particle concentration, are dispersed. The soliton amplitude has a finite-time singularity, but this is ultimately unimportant, as when the soliton grows too large, the assumptions made in deriving the KdV equation (with weak amplification) in the first place no longer hold. Nevertheless, the finite-time singularity may be taken to indicate that the initially weakly nonlinear disturbance eventually becomes fully nonlinear.

As time goes on, KdV remains the leading-order equation, but the shelf and higher-order nonlinear terms reorder themselves in the asymptotic expansion. This leads to several changes in the secularity condition and thus alters the time dependence of the soliton amplitude, weakening the finite-time singularity. Eventually, all the nonlinear terms become of the same order and there is a fundamental change in the solution. A new equation is necessary, and it turns out to be quasi-stationary but fully nonlinear and dispersive.

This new fully nonlinear equation has a solitary wave solution and the velocity of this can be expressed in terms of the amplitude of the wave. If we allow this velocity to vary on a slow timescale, then an expression can be written down for the secularity condition required to ensure non-resonant terms at the next order. This is shown to reduce to the previous secularity condition obtained for the growing KdV soliton when we take a small-amplitude limit for the solitary wave, and the 'sech-squared' shape can also be recovered. Thus the soliton and solitary wave can be matched directly without the need for an intermediate time-dependent region. Further, it is shown that although the solitary wave is still being amplified initially, it eventually tends to a limiting size. This maximum amplitude (i.e. ultimate voidage within the solitary wave slug) only depends on n and ϕ_0 and there is no longer a finite-time singularity.

The presence of the shelf behind the KdV soliton indicates that there is a region with a higher particle concentration left in the wake of the developing void. In terms of soliton theory, this shelf is needed to conserve mass and here it occurs because some of the particles have to be displaced from the void region and must end up in the uniformly fluidized part of the bed below the void. The shelf does not grow to become $O(1)$ in amplitude but it does grow in length, contributing an $O(1)$ amount to the mass balance. Thus instead of a small region behind the soliton becoming very concentrated and defluidized, a large portion of the bed is left with a higher particle concentration than before. Thus, it appears that the voidage waves remove some of the excess fluid which is not required to maintain uniform fluidization.

The model appears appropriate for describing the propagation and evolution of voidage waves due to perturbations in the bed. There is some experimental evidence that it is the breakup of such waves that leads to bubble formation in a two-dimensional bed, but that these structures persist in narrow tubes. Thus, it is plausible to interpret the behaviour as the second type of slugging, where there are alternating horizontal bands of high and low voidage with particle raining at the interfaces. So one solitary wave corresponds to one void and we need a succession of such waves. It is possible that the oscillatory tail attached to the back of the shelf may help induce further bands of void and particles to form lower down the bed, thereby giving the periodic slugging behaviour which is observed.

Our approach has successfully described the initial growth of an arbitrarily small disturbance (of large enough spatial length so that the Froude number is small) first into a soliton-like voidage perturbation in the bed and then through all the intermediate regions until it has developed into a fully nonlinear planar wave, with a well-defined maximum size. This has not been done before for any of the fluidized bed models in the literature and we believe that it represents a significant step forward in the understanding of slug formation. It also represents, we believe, an unusual contribution to nonlinear wave theory, which usually concentrates on either the weakly nonlinear limit or the fully nonlinear case for travelling waves, without showing the continuous development, through a hierarchy of recognizable intermediate stages, from one initial situation to the final one.

Since this work was completed, the authors have had sight of the forthcoming paper by Hayakawa, Komatsu & Tsuzuki (1993). This paper identifies KdV as the relevant equation in the weakly nonlinear regime, but does not attempt to describe the evolution of faint KdV voidage solutions into large-amplitude voidage slugs, as has been done in the present paper. We reject the implication in §5 of Hayakawa *et al.* (1993) that our 'correction to KdV is, however, not correct'.

One of the authors (S. E. H.) gratefully acknowledges the support of a Research Studentship from the SERC and of a Research Fellowship at Sidney Sussex College, Cambridge for the period in which this research was carried out.

Appendix A. Stability for all wavenumbers

The full linearized equation (3.2) can be written

$$\phi'_\tau+\gamma_0\phi'_{XXX}+(\alpha_0^2-P_0)\phi_0\phi'_{XX}-2F^2\phi_0\alpha_0\phi'_{\tau X}-F^2\frac{\gamma_0}{\alpha_0}\phi'_{\tau XX}+F^4\phi_0\phi'_{\tau\tau}=0, \quad \text{(A 1)}$$

using the transformed coordinates given in (3.12). We can then investigate the linear stability of this without making any assumptions about the size of the Froude number. Firstly, we set

$$\phi' = A\,e^{i(kX-\omega\tau)}, \tag{A 2}$$

where the complex frequency ω is given by

$$\omega = \omega_r + i\omega_i, \tag{A 3}$$

and substitute (A 2) into (A 1). Equating the real and imaginary parts yields two simultaneous equations which can then be solved to give

$$\omega_r = -\frac{\alpha_0 k}{F^2} \pm \frac{1}{2\sqrt{2F^4\phi_0}}[4F^4P_0\phi_0^2k^2-\mu^2+\{(4F^4P_0\phi_0^2k^2-\mu^2)^2+16F^4\alpha_0^2\phi_0^2k^2\}^{\frac{1}{2}}]^{\frac{1}{2}}, \tag{A 4}$$

$$\omega_i = \frac{1}{2F^4\phi_0}\left(-\mu \pm \frac{1}{\sqrt{2}}[\mu^2-4F^4P_0\phi_0^2k^2+\{(4F^4P_0\phi_0^2k^2-\mu^2)^2+16F^4\alpha_0^2\phi_0^2k^2\}^{\frac{1}{2}}]^{\frac{1}{2}}\right), \tag{A 5}$$

where

$$\mu = 1+F^2k^2\gamma_0/\alpha_0. \tag{A 6}$$

The neutral stability curve is obtained by putting $\omega_i = 0$, which gives

$$P_0 = \alpha_0^2/(1+F^2k^2\gamma_0/\alpha_0)^2, \tag{A 7}$$

as indicated by Needham & Merkin (1983), and along this curve

$$\omega_r = -\gamma_0 k^3(1+F^2k^2\gamma_0/\alpha_0)^{-1}. \tag{A 8}$$

If $P_0 < \alpha_0^2$, then we can write

$$P_0 = \alpha_0^2/(1+F^2k_c^2\gamma_0/\alpha_0)^2, \tag{A 9}$$

for some k_c. Thus the normal modes with wavenumber in the range $0 < k \leqslant k_c$ are unstable and those with $k > k_c$ are stable, so waves with a long wavelength grow but those with small wavelengths decay. If we set

$$P_0 = \alpha_0^2 - \Delta, \tag{A 10}$$

then

$$k_c = \frac{1}{F}\left(\frac{\alpha_0}{\gamma_0}\right)^{\frac{1}{2}}\left[\left(1-\frac{\Delta}{\alpha_0^2}\right)^{-\frac{1}{2}}-1\right]^{\frac{1}{2}}, \tag{A 11}$$

and so if Δ is small,

$$k_c = \frac{1}{F}\left\{\left(\frac{\Delta}{2\gamma_0\alpha_0}\right)^{\frac{1}{2}}+O(\Delta)\right\}. \tag{A 12}$$

Expanding ω_r and ω_i in powers of k from (A 9) and (A 10) for the positive root gives

$$\omega_r = -\gamma_0 k^3 - 2F^2\phi_0^2\alpha_0\Delta k^3 + O(k^4), \tag{A 13}$$

$$\omega_i = \phi_0\Delta k^2 + F^2\phi_0\left(-2\gamma_0\alpha_0+\frac{\gamma_0}{\alpha_0}\Delta+O(\Delta^2)\right)k^4+O(k^6), \tag{A 14}$$

and these are valid for any F and for $k, \Delta \ll 1$. If we now fix $\Delta = O(F)$ and $F \ll 1$, then for $k \ll k_c = O(F^{-\frac{1}{2}})$, we have

$$\phi' = \exp[i(kX+\gamma_0 k^3\tau)+\phi_0\Delta k^2\tau], \tag{A 15}$$

which corresponds to the equation

$$\phi'_\tau + \gamma_0\phi'_{XXX} = -\phi_0\Delta\phi'_{XX}. \tag{A 16}$$

Thus we have the linearized KdV equation with an amplifying Burgers-type perturbation term. If $k \sim k_c$, then we have

$$\phi' = \exp\left[\mathrm{i}(kX+\gamma_0 k^3\tau)+\phi_0(\Delta-2\gamma_0\alpha_0 F^2k^2)k^2\tau\right], \tag{A 17}$$

so that the correct form of the differential equation is

$$\phi'_\tau+\gamma_0\phi'_{XXX} = -\phi_0(\Delta\phi'_{XX}+2\gamma_0\alpha_0 F^2\phi'_{XXXX}). \tag{A 18}$$

Therefore close to the stability boundary, the fourth-order derivative becomes important and must also be retained. This has a stabilizing influence and prevents the rapid growth of short-scale modes. There is a maximally amplified wavenumber $k = k_a$ and, accordingly, if $\phi'(X,0)$ is the initial distribution, then the distribution that evolves over long times in the linear regime is a wavepacket, of carrier wavenumber k_a, amplitude $\tilde{\phi}'(k_a,0)$, the spatial Fourier transform of $\phi'(X,0)$, and extent increasing linearly with t. This evolved distribution then serves to determine the nonlinear breakup into solitons under KdV.

Appendix B. Magma equations

Here we analyse a special case of the fluidized bed system which is valid for $\phi_0 \ll 1$. The leading-order equations have also been derived for magma migration (Barcilon & Richter 1986). We begin by rescaling the original system of equations (2.12) and (2.13) using

$$\phi = \phi_0\tilde{\phi}, \quad v = \phi_0\tilde{v}, \tag{B 1}$$

in addition to

$$\begin{gathered}\tilde{F}^2 = F^2\phi_0, \quad \alpha_0 = n+1, \quad X = x-\alpha_0 t,\\ \tilde{\phi} = 1+\tilde{F}^2\phi', \quad \tilde{v} = \tilde{F}^2 v', \quad \tau = \tilde{F}^2 t.\end{gathered} \tag{B 2}$$

Then neglecting terms $O(\phi_0)$ yields the equation

$$\phi'_\tau+\beta_0\phi'\phi'_X+\gamma_0\phi'_{XXX} = -\tilde{F}\delta_0\phi'_{XX}+O(\tilde{F}^2), \tag{B 3}$$

where now

$$\beta_0 = n(n+1), \quad \gamma_0 = (n+1)/R, \quad \tilde{F}\delta_0 = \alpha_0^2-P_0, \tag{B 4}$$

and we choose $p'_s(\tilde{\phi}) = -P_0$ for all $\tilde{\phi}$. Thus, we have the KdV equation with a Burgers perturbation term again and we can investigate the growth of a single soliton solution as before. The slow variation of the soliton now takes place on a timescale $T = \tilde{F}\tau$.

The fully nonlinear equations can be found using (B 1) together with

$$\zeta = \tilde{F}^{-1}(x-x_0-\hat{\xi}), \quad T-T_0 = \tilde{F}^2\hat{T}, \quad \hat{T} = \tilde{F}T', \tag{B 5}$$

to obtain

$$c\tilde{\phi}_\zeta+\tilde{v}_\zeta = \tilde{F}\tilde{\phi}_{T'}, \tag{B 6}$$

$$\tilde{\phi}^{-(n+1)}(1-\tilde{v})-1+\tilde{v}_{\zeta\zeta}/R = \tilde{F}[c^2-(n+1)^2]\tilde{\phi}_\zeta, \tag{B 7}$$

in place of (5.30) and (5.4). If we set

$$\tilde{\phi} = \tilde{\phi}_0+\tilde{F}\tilde{\phi}_1+\cdots, \quad \tilde{v} = \tilde{v}_0+\tilde{F}\tilde{v}_1+\cdots, \tag{B 8}$$

then the results analogous to (5.8) and (5.10) for the leading-order solutions are

$$\tilde{v}_0 = c(1-\tilde{\phi}_0), \tag{B 9}$$

$$\frac{c}{2R}\tilde{\phi}^2_{0\zeta} = (1-\tilde{\phi}_0)+\frac{(1-c)}{n}(1-\tilde{\phi}_0^{-n})+c(1-\tilde{\phi}_0^{-(n-1)}). \tag{B 10}$$

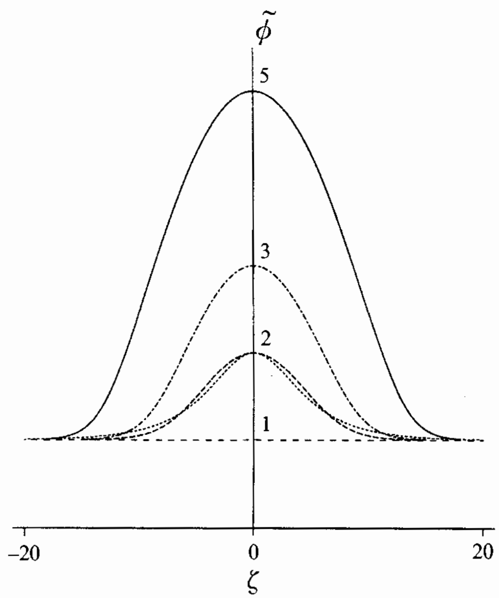

FIGURE 4. Solitary wave solutions for $\phi_0 \ll 1$; ——, $A = 5$; –·–·–, $A = 3$; –––, $A = 2$. Sech-squared soliton: · · · ·, $A = 2$.

Equation (B 10) can be rewritten as

$$\frac{c}{2R}\tilde{\phi}_{0\zeta}^2 = (c-\Gamma(\tilde{\phi}_0;n))\left[\frac{\tilde{\phi}_0^{n-1}-1}{(n-1)\,\tilde{\phi}_0^{n-1}}-\frac{\tilde{\phi}_0^n-1}{n\tilde{\phi}_0^n}\right], \tag{B 11}$$

where

$$\Gamma(\tilde{\phi}_0;n) = (n-1)\left[\frac{\tilde{\phi}_0^n(\tilde{\phi}_0-1)+(1-\tilde{\phi}_0^n)}{n\tilde{\phi}_0^{n-1}(\tilde{\phi}_0-1)-(n-1)(\tilde{\phi}_0^n-1)}\right]. \tag{B 12}$$

But the velocity c is given by $c = \Gamma(A;n)$ where A is the amplitude of the wave, so we have

$$c = (n-1)\left[\frac{nA^{n+1}-(n+1)\,A^n+1}{A^n-nA+(n-1)}\right]. \tag{B 13}$$

If we take $n = 2$, then (B 13) reduces to the simple form

$$c = 2A+1. \tag{B 14}$$

In this case, we can integrate (B 10) to give an implicit expression for the profile $\tilde{\phi}_0(\zeta)$,

$$\zeta = \pm\left(\frac{2A+1}{2R}\right)^{\frac{1}{2}}\left\{2(A-\tilde{\phi}_0)^{\frac{1}{2}}-\frac{1}{(A-1)^{\frac{1}{2}}}\ln\left[\frac{(A-1)^{\frac{1}{2}}-(A-\tilde{\phi}_0)^{\frac{1}{2}}}{(A-1)^{\frac{1}{2}}+(A-\tilde{\phi}_0)^{\frac{1}{2}}}\right]\right\}. \tag{B 15}$$

If we put $R = 1$, this is the same result as that obtained by Barcilon & Richter (1986) for their $n = 3$, $\phi_0 \ll 1$ case for magma flows. The solitary wave profiles for this are illustrated in figure 4, along with a comparison with a 'sech-squared' soliton solution for small amplitude.

To find the amplitude variation with time, we need to consider higher-order terms. The equations for $\tilde{\phi}_1$ and $\tilde{v}_1$ are given by

$$c\tilde{\phi}_{1\zeta}+\tilde{v}_{1\zeta} = \tilde{\phi}_{0T'}, \tag{B 16}$$

$$-(n+1)\frac{\tilde{\phi}_1}{\tilde{\phi}_0^{n+2}}(1-\tilde{v}_1)-\frac{\tilde{v}_1}{\tilde{\phi}_0^{n+1}}+\frac{\tilde{v}_{1\zeta\zeta}}{R} = [c^2-(n+1)^2]\,\tilde{\phi}_{0\zeta}. \tag{B 17}$$

The adjoint transformation can be found as before and a solution to the homogeneous problem is given by

$$l_1 = (1-c)(1-\tilde{\phi}^{-(n+1)}) + \frac{c(n+1)}{n}(1-\tilde{\phi}_0^{-n}), \quad l_2 = -c\tilde{\phi}_{0\zeta}. \tag{B 18}$$

Thus the secularity condition (5.35) now becomes

$$\int_{-\infty}^{\infty}\left[(1-c)(1-\tilde{\phi}_0^{-(n+1)}) + \frac{c(n+1)}{n}(1-\tilde{\phi}_0^{-n})\right]\tilde{\phi}_{0T'}\,\mathrm{d}\zeta - \int_{-\infty}^{\infty} c[c^2-(n+1)^2]\,\tilde{\phi}_{0\zeta}^2\,\mathrm{d}\zeta = 0. \tag{B 19}$$

We can rewrite this as

$$\mathrm{d}I/\mathrm{d}T' = J, \tag{B 20}$$

where

$$J = \int_{-\infty}^{\infty} c[c^2-(n+1)^2]\,\tilde{\phi}_{0\zeta}^2\,\mathrm{d}\zeta \tag{B 21}$$

and

$$I = \int_{-\infty}^{\infty}\left[\frac{n+c}{n}\tilde{\phi}_0 + \frac{1-c}{n}\tilde{\phi}_0^{-n} + \frac{c(n+1)}{n(n-1)}\tilde{\phi}_0^{-(n-1)} + K\right]\mathrm{d}\zeta. \tag{B 22}$$

If we choose K so that the integrand vanishes when $\tilde{\phi}_0 = 1$, which corresponds to the state of uniform fluidization, and restrict ourselves to the $n = 2$ case where

$$\tilde{\phi}_{0\zeta} = \left(\frac{2R}{2A+1}\right)^{\frac{1}{2}} \frac{(A-\tilde{\phi}_0)^{\frac{1}{2}}(\tilde{\phi}_0-1)}{\tilde{\phi}_0} \tag{B 23}$$

from (B 10), then

$$I = \frac{4}{3}\left(\frac{2A+1}{2R}\right)^{\frac{1}{2}}\{(2A^2-2A-3)(A-1)^{\frac{1}{2}} + 3A^{\frac{1}{2}}\ln[A^{\frac{1}{2}}+(A-1)^{\frac{1}{2}}]\}. \tag{B 24}$$

Similarly, we have

$$J = \tfrac{16}{3}(A+2)(A-1)(2A+1)^{\frac{1}{2}}(2R)^{\frac{1}{2}}\{(A-1)^{\frac{1}{2}}(A+2) - 3A^{\frac{1}{2}}\ln[A^{\frac{1}{2}}+(A-1)^{\frac{1}{2}}]\}, \tag{B 25}$$

so

$$\mathrm{d}A/\mathrm{d}T' = 8(A+2)(A-1)(2A+1)A^{\frac{1}{2}}R \times\left[\frac{(A-1)^{\frac{1}{2}}(A+2) - 3A^{\frac{1}{2}}\ln[A^{\frac{1}{2}}+(A-1)^{\frac{1}{2}}]}{2A^{\frac{1}{2}}(A-1)^{\frac{1}{2}}[12A^2-A-5] + 3(4A+1)\ln[A^{\frac{1}{2}}+(A-1)^{\frac{1}{2}}]}\right]. \tag{B 26}$$

If we let $A \to 1 + \tilde{F}^{2-2a}12\gamma_0\kappa^2/\beta_0$ and $\mathrm{d}/\mathrm{d}T' \to \tilde{F}^{4-4a}\,\mathrm{d}/\mathrm{d}T'$ and just retain the leading-order behaviour, then (B 26) reduces to the expected result for the final soliton region given by (4.62), with $d = (64/15)\,\alpha_0\gamma_0$ being appropriate for the $\phi_0 \ll 1$ situation. There are no roots of the right-hand side of (B 26) for $\tilde{\phi} > \phi_0$, i.e. $A > 1$ and so there is no possibility of the amplitude tending to a limiting value, but this is an artefact of the scalings used in (B 1) and so we expect $A \to \infty$.

REFERENCES

Ablowitz, M. J. & Segur, H. 1981 *Solitons and the Inverse Scattering Transform*. SIAM.

Auzerais, F. M., Jackson, R. & Russel, W. B. 1988 The resolution of shocks and the effects of compressible sediments in transient settling. *J. Fluid Mech.* **195**, 437–462.

Barcilon, V. & Lovera, O. M. 1989 Solitary waves in magma dynamics. *J. Fluid Mech.* **204**, 121–133.

Barcilon, V. & Richter, F. M. 1986 Nonlinear waves in compacting media. *J. Fluid Mech.* **164**, 429–448.

Batchelor, G. K. 1988 A new theory of the instability of a uniform fluidized bed. *J. Fluid Mech.* **193**, 75–110.

Couderc, J.-P. 1985 Incipient fluidization and particulate systems. In *Fluidization*, 2nd edn. (ed. J. F. Davidson, R. Clift & D. Harrison), pp. 1–46. Academic.

Courant, R. & Hilbert, D. 1962 *Methods of Mathematical Physics*, Vol. II. Wiley-Interscience.

Crighton, D. G. 1991 Nonlinear waves in fluidized beds. In *Nonlinear Waves in Real Fluids* (ed. A. Kluwick), pp. 83–90. Springer.

Davidson, J. F. & Harrison, D. 1963 *Fluidised Particles*. Cambridge University Press.

Didwania, A. & Homsy, G. M. 1981 Flow regimes and flow transitions in liquid-fluidized beds. *Intl J. Multiphase Flow* **7**, 563–580.

Drazin, P. G. & Johnson, R. S. 1989 *Solitons: an Introduction*. Cambridge University Press.

Drew, D. A. & Segal, L. A. 1971 Averaged equations for two-phase flows. *Stud. Appl. Maths* **50**, 205–231.

El-Kaissy, M. M. & Homsy, G. M. 1976 Instability waves and the origin of bubbles in fluidized beds. Part I: experiments. *Intl J. Multiphase Flow* **2**, 379–395.

Fanucci, J. B., Ness, N. & Yen, R.-H. 1979 On the formation of bubbles in gas–particulate fluidized beds. *J. Fluid Mech.* **94**, 353–367.

Foscolo, P. U. & Gibilaro, L. G. 1984 A fully predictive criterion for the transition between particulate and aggregate fluidization. *Chem. Engng Sci.* **39.2**, 1667–1675.

Foscolo, P. U. & Gibilaro, L. G. 1987 Fluid dynamic stability of fluidized suspensions: the particle bed model. *Chem. Engng Sci.* **42.1**, 1489–1500.

Garg, S. K. & Pritchett, J. W. 1975 Dynamics of gas-fluidized beds. *J. Appl. Phys.* **46**, 4493–4500.

Geldart, D. 1973 Types of gas fluidization. *Powder Technol.* **7**, 285–292.

Göz, M. F. 1992 On the origin of wave patterns in fluidized beds. *J. Fluid Mech.* **240**, 379–404.

Harris, S. E. 1992 Nonlinear wave equations with dispersion, dissipation and amplification. PhD dissertation, University of Cambridge.

Hayakawa, H., Komatsu, T. S. & Tsuzuki, T. 1993 Pseudo solitons in fluidized beds. *Physica* A (in press).

Homsy, G. M., El-Kaissy, M. M. & Didwania, A. 1980 Instability waves and the origin of bubbles in fluidized beds. Part II. *Intl J. Multiphase Flow* **6**, 305–318.

Jackson, R. 1963*a* The mechanics of fluidised beds. Part I. The stability of the state of uniform fluidisation. *Trans. Instn Chem. Engrs* **41**, 13–21.

Jackson, R. 1963*b* The mechanics of fluidised beds. Part II. The motion of fully developed bubbles. *Trans. Instn Chem. Engrs* **41**, 22–28.

Jackson, R. 1971 Fluid Mechanical Theory. In *Fluidization* (ed. J. F. Davidson & D. Harrison), pp. 65–119. Academic.

Johnson, R. S. 1972 Shallow water waves on a viscous fluid – the undular bore. *Phys. Fluids* **15**, 1693–1699.

Karpman, V. I. & Maslov, E. M. 1977 Perturbation theory for solitons. *Sov. Phys. JETP* **46/2**, 281–291.

Karpman, V. I. & Maslov, E. M. 1978 Structure of tails produced under the action of perturbations on solitons. *Sov. Phys. JETP* **48/2**, 252–259.

Kluwick, A. 1983 Small-amplitude finite-rate waves in suspensions of particles in fluids. *Z. Angew. Math. Mech.* **63**, 161–171.

Kluwick, A. 1991 Weakly nonlinear kinematic waves in suspensions of particles in fluids. *Acta Mechanica* **88**, 205–217.

Knickerbocker C. J. & Newell, A. C. 1980 Shelves and the Korteweg–de Vries equation. *J. Fluid Mech.* **98**, 803–818.

Knickerbocker, C. J. & Newell, A. C. 1985 Reflections from solitary waves in channels of decreasing depth. *J. Fluid Mech.* **153**, 1–16.

Komatsu, T. S. & Hayakawa, H. 1993 Nonlinear waves in fluidized beds. *Phys. Lett.* A (submitted).

KURDYUMOV, V. N. & SERGEEV, YU. A. 1987 Propagation of nonlinear waves in a fluidized bed in the presence of interaction between the particles of the dispersed phase. *Fluid Dyn.* **22/2**, 235–242.

KUZNETSOV, V. V., NAKORYAKOV, V. E., POKUSAEV, B. G. & SHREIBER, I. R. 1978 Propagation of perturbations in a gas–liquid mixture. *J. Fluid Mech.* **85**, 85–96.

LEIBOVICH, S. & SEEBASS, A. R. 1974 *Nonlinear Waves.* Cornell University Press.

LEVA, M. 1959 *Fluidization.* McGraw-Hill.

LIU, J. T. C. 1982 Note on a wave-hierarchy interpretation of fluidized bed instabilities. *Proc. R. Soc. Lond.* A **380**, 229–239.

LIU, J. T. C. 1983 Nonlinear unstable wave disturbances in fluidized beds. *Proc. R. Soc. Lond.* A **389**, 331–347.

MURRAY, J. D. 1965 On the mathematics of fluidization. *J. Fluid Mech.* **21**, 465–493.

NAKAYAMA, M. & MASON, D. P. 1992 Rarefactive solitary waves in two-phase fluid flow of compacting media. *Wave Motion* **15**, 357–392.

NAYFEH, A. H. 1973 *Perturbation Methods.* Wiley-Interscience.

NEEDHAM, D. J. & MERKIN, J. H. 1983 The propagation of a voidage disturbance in a uniformly fluidized bed. *J. Fluid Mech.* **131**, 427–454.

NEEDHAM, D. J. & MERKIN, J. H. 1984 The evolution of a two-dimensional small-amplitude voidage disturbance in a uniformly fluidized bed. *J. Engng Maths* **18**, 119–132.

NEEDHAM, D. J. & MERKIN, J. H. 1986 The existence and stability of quasi-steady periodic voidage waves in a fluidized bed. *Z. Angew. Math. Phys.* **37**, 322–339.

RICHARDSON, J. F. 1971 Incipient fluidization and particulate systems. In *Fluidization* (ed. J. F. Davidson & D. Harrison), pp. 26–64. Academic.

SASA, S.-I. & HAYAKAWA, H. 1992 Void-fraction dynamics in fluidization. *Europhys. Lett.* **17**, 685–689.

SCHÜGERL, K. 1971 Rheological behaviour of fluidized systems. In *Fluidization* (ed. J. F. Davidson & D. Harrison), pp. 261–292. Academic.

VERLOOP, J. & HEERTJES, P. M. 1970 Shock waves as a criterion for the transition from homogeneous to heterogeneous fluidization. *Chem. Engng Sci.* **25**, 825–832.

ZENZ, F. A. 1971 Regimes of fluidized behaviour. In *Fluidization* (ed. J. F. Davidson & D. Harrison), pp. 1–23. Academic.